# Quantum Field Theory I: PHYS 721
# Exam

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## Overview

This exam is due on **Tuesday December 21** at **5 pm**. Submit your manuscript either as a PDF via email to me, or by hand to me, or to my inbox in the main office[1] and include your name in the filename of your submission. **Late submissions will not be accepted.**

Read the following carefully.

This exam will be graded out of 60 points. You are free to use textbooks and online resources, but you must cite all references that you use. You may discuss the exam in general terms with other students in the class, particularly to clarify aspects of specific parts of the problems, but the solutions must be your own work and you may not share solutions. You can, for example,

1. talk about general strategies;
2. explain your understanding of what you think the question is asking for; or
3. point out a useful reference that helped you solve something.

But you cannot, for example:

1. discuss your approach to a given part of a problem or give explicit solutions;
2. write solutions for each other;
3. share solutions with each other; or
4. proof read any part of another person's solutions.

You are welcome to discuss the exam with me.

This exam will:

- introduce you to Fermi's theory of the weak interaction and the concept of effective theories;
- help you solidify your understanding of scattering in field theories involving fermions;
- provide an opportunity to apply the concepts and techniques of quantum field theory to an unfamiliar theory: the weak nuclear force.

The exam is designed to help you demonstrate that you can apply your knowledge of quantum field theory techniques to an unfamiliar example to construct new results, and to draw conclusions based on those results. Being able to apply techniques is an example of "procedural knowledge", which demonstrates a deeper understanding of the material than "factual knowledge". The educational basis for this is known as Bloom's revised taxonomy[2], which divides learning into knowledge dimensions (factual, conceptual, procedural, and meta-cognitive) and cognitive process dimensions (understand, analyse, apply, evaluate, and create). Generally speaking, effectively using a cognitive process to learn requires the ability to use the

[1] Please email me to confirm I have received your submission if you choose the inbox option.

[2] For more details on Bloom's revised taxonomy, see, for example https://cft.vanderbilt.edu/guides-sub-pages/blooms-taxonomy/ or https://www.coloradocollege.edu/other/assessment/how-to-assess-learning/learning-outcomes/blooms-revised-taxonomy.html, or the original and more complete discussion of the revisions in Anderson and Krathwohl, *A Taxonomy for Learning, Teaching, and Assessing: A Revision of Bloom's Taxonomy of Educational Objectives* (2001). Addison Wesley Longman, New York.

preceding cognitive process in that list. For example, to be able to analyse, you first have to understand. Reproducing material from lecture notes corresponds to factual knowledge and the "understand" cognitive process. Applying techniques from class and calculating new results represents procedural knowledge implemented using the "apply" cognitive process. Drawing conclusions from your results is procedural knowledge demonstrated through the "evaluate" cognitive process.

## Motivation

In this exam we will study muon decay in the standard model of particle physics. This decay is mediated by the weak force and, to motivate our understanding of this new theory, we will start by briefly considering a toy model.

In class I said that we did not need to consider four-fermion interactions, that is, interaction terms of the form $(\overline{\psi}\psi)^2$. This, however, applies only to renormalisable theories. These sort of interaction terms do appear in *effective field theories*, which are low-energy descriptions that use only the *relevant degrees of freedom* at a specific energy scale or range of energy scales. At low energies, we cannot tell what is happening at very high energies (or short-distances), and we parameterise our ignorance of the true nature of reality at short distances by using effective theories. These are only valid up to a certain momentum cutoff (i.e. down to a certain length scale), and they need not be renormalisable. Fermi's theory of the weak force is the classic example of this.

In Section 1 we look at a toy model of this theory to begin to understand how these ideas work, using two different model Lagrangians. In Section 2 we then consider muon decay $\mu^- \to e^- \overline{\nu}_e \nu_\mu$, introducing only the necessary ingredients of the weak force, which mediates this process. A **full understanding of the weak force is not required to answer these questions** and our focus is on applying our knowledge of scattering to a new theory. In Section 3, we consider Fermi's effective theory of the weak interaction to derive the invariant matrix element for muon decay, which we compare to the standard model result from Section 2. In Section 4 we choose specific kinematics for muon decay to extract a leading order prediction for the Fermi coupling constant, which parameterises the strength of the weak interaction.

## Section 1 [15]

(a) Consider the Lagrangian of a free Dirac fermion $\Psi$, with a four-fermion interaction term

$$\mathcal{L}_T = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + G(\overline{\Psi}\Psi)^2,$$

where $G$ is a coupling constant. Draw the leading-order diagram for fermion-antifermion scattering ($\Psi\overline{\Psi} \to \Psi\overline{\Psi}$) in this theory and write down the corresponding invariant matrix element.

(b) Consider the Yukawa Lagrangian of a Dirac fermion $\Psi$, coupled to a massive scalar field, $\phi$,

$$\mathcal{L}_Y = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - M^2\phi^2\right) - g\phi\overline{\Psi}\Psi.$$

i) Draw the leading-order diagram for fermion-antifermion scattering ($\Psi\overline{\Psi} \to \Psi\overline{\Psi}$) in the $s$-channel in this theory and write down the corresponding invariant matrix element.

ii) Work in the centre-of-momentum frame, in which the sum of the incoming momentum is given by $(p_1 + p_2)^2 = 4E^2$, and show that at low energies the matrix elements from part (a) and (b.i) are the same, provided $G \propto g^2/M^2$. Find the constant of proportionality.

iii) Discuss how the results from (a) and (b.ii) are related.

**Section 2** **[10]**

Now that we have seen a toy example of how an effective theory can represent a scattering process, we will turn to a more realistic example and look at muon decay in the standard model. You do not need to have a full understanding of the weak force to answer the following questions!

In the standard model of particle physics we have two types of fermions: quarks and leptons. There are three generations of each of these types of fermions. The leptons are electrons, muons, and taus, and their neutrino partners $\nu_e$, $\nu_\mu$, and $\nu_\tau$. The quark families are up and down, strange and charm, and top and bottom. Neglecting neutrino masses, which are much lighter than other scales in the standard model, means that only left-handed neutrinos couple to the standard model[3]. The other leptons, however, can be both left- or right-handed, and we denote the Weyl spinors representing the left- and right-handed electrons, for example, as

$$e_L = \frac{1-\gamma^5}{2}e = P_L e, \qquad e_R = \frac{1+\gamma^5}{2}e = P_R e,$$

where $e$ is a four-component Dirac spinor. The neutrino spinors can be treated in the same way, except that there are only left-handed neutrinos.

The weak force is mediated by the charged $W_\mu^+$ and $W_\mu^-$ and neutral $Z_\mu^0$ massive vector bosons. You can treat these as analogous to the photon in QED, except they have charge and mass. The charge only enters the problem through the coupling to the fermions, but the mass appears in the propagator. At leading order, only the charged vector bosons take part in the weak decay of the muon, and they have a propagator given by

$$\widetilde{D}_{\mu\nu}(q) = \frac{-i}{q^2 - m_W^2 + i\epsilon}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_W^2}\right).$$

Here $m_W$ is the mass of the $W_\mu^\pm$ bosons, which is $m_W \sim 80.4$ GeV. If all the masses of the initial and final particles are much less than this mass, then we can approximate the propagator as

$$\widetilde{D}_{\mu\nu}(q) \simeq \frac{i g_{\mu\nu}}{m_W^2}.$$

In the following questions, you may assume that the neutrinos are massless, and that the masses of the electron and muon are much smaller than the mass of the $W_\mu^-$ boson,

$$\frac{m_e^2}{m_W^2} \ll 1, \qquad \frac{m_\mu^2}{m_W^2} \ll 1.$$

For muon decay, the relevant interaction between the $W_\mu^-$ bosons and the electron and muon lepton families is given by the interaction Lagrangian

$$\mathcal{L}_{\text{int.}} = \frac{g}{\sqrt{2}}\left(\overline{e}_L \gamma^\mu \nu_{e,L} + \overline{\nu}_{\mu,L}\gamma^\mu \mu_L\right) W_\mu.$$

[3] The full explanation of this requires a fuller understanding of the weak force [see, for example, Chapter 29 of Schwartz [1]], which is not necessary here, and you can take this as a given fact.

This origin of this interaction term is analogous to the way we coupled the conserved vector current to the photon in QED. Here, instead, we couple the gauge boson (the massive charged $W_\mu^\pm$ boson) to the corresponding currents.

(a) Draw the leading order Feynman diagram for muon decay

$$\mu^- \to \nu_\mu \overline{\nu}_e e^-.$$

[Hint: this involves two different vertices that connect a lepton, its neutrino partner and a $W$ boson. Be careful with which states are initial and final states.]

(b) Use the properties of the gamma matrices to show that the interaction, $\mathcal{L}_{\text{int.}}$, can be written in "$V-A$" form

$$\mathcal{L}_{\text{int.}} = \frac{g}{\sqrt{2}} \left( \bar{e} \gamma^\mu P_L \nu_e + \overline{\nu}_\mu \gamma^\mu P_L \mu \right) W_\mu.$$

Note that here the spinors are now four-component Dirac spinors, and because the neutrino only occurs as a left-handed particle, its four-component spinor obeys

$$P_L \nu = \nu.$$

The nomenclature "$V-A$" comes from the fact that this interaction term, $\mathcal{L}_{\text{int.}}$, looks like the vector current minus the axial-vector current.

(c) Write down the invariant matrix element for the leading contribution to muon decay (i.e. the Feynman diagram you drew in part (a) of this Section).

**Section 3** **[10]**

Consider now Fermi's effective four-fermion interaction Lagrangian

$$\mathcal{L}_F = -\frac{G_F}{\sqrt{2}} \left[ \overline{\nu}_\mu \gamma^\mu (1-\gamma^5) \mu \right] \left[ \bar{e} \gamma_\mu (1-\gamma^5) \nu_e \right].$$

The strength of the interaction is given by $G_F$, known as the "Fermi coupling constant".

(a) Draw the Feynman diagram that represents the leading order contribution to muon decay with this interaction Lagrangian.

(b) Write down the corresponding invariant matrix element for this diagram.

(c) Compare your expressions for the invariant matrix element in the full theory [your solution to Section 2, question (c)] and the effective theory [your solution to Section 3, question (b)] and show that these expressions are the same provided

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 m_W^2}.$$

**Section 4** **[25]**

(a) Show that the invariant matrix element squared, in the effective theory, can be written as

$$|\mathcal{M}|^2 = 64 G_F^2 (p_\mu q_1^\mu)(k_\nu q_2^\nu),$$

where $p^\mu$ is the initial momentum of the muon, $k^\mu$ is the momentum of the final state electron, and $q_{1,2}$ are the momenta of the final state electron and muon neutrinos, respectively.

(b) Work in the muon rest frame and use the expression for the differential decay rate to three final state particles,

$$\mathrm{d}\Gamma(A \to p_1, p_2, p_3) = \frac{1}{2m_A}\left(\prod_{f=1}^{3} \frac{\mathrm{d}^3\vec{p}_f}{(2\pi)^3 2E_{p_f}}\right) |\mathcal{M}(A \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}\left(k_A - \sum_{f=1}^{3} p_f\right)$$

to show that, in the limit in which we can neglect the electron mass,

$$\mathrm{d}\Gamma = \frac{G_F^2}{12\pi^3 m_\mu}\Big(q^2(p\cdot k) + 2(q\cdot p)(q\cdot k)\Big)\, E\,\mathrm{d}E,$$

where $q^\mu = q_1^\mu + q_2^\mu$, and $E = E_k$ is the energy of the final state electron. You may find the integral

$$I^{\mu\nu}(q^2) = \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E_2}\delta^{(4)}(q - q_1 - q_2) q_1^\mu q_2^\nu = \frac{1}{6\pi(4\pi)^4}\Big(q^2 g^{\mu\nu} + 2q^\mu q^\nu\Big)$$

helpful.

(c) The kinematics of three-body final states means that the minimum final state electron energy is $E_{\min} = m_e \simeq 0$ and the maximum is $E_{\max} = m_\mu/2$. Integrate the differential decay rate over this kinematic region to show the total decay rate is

$$\Gamma \simeq \frac{G_F^2 m_\mu^5}{192\pi^3}.$$

(d) Use the muon lifetime and mass [4]

$$\tau_\mu = (2.1969811 \pm 0.0000022)\times 10^{-6}\ \mathrm{s}, \qquad m_\mu = (105.6583745 \pm 0.0000024)\ \mathrm{MeV}$$

to obtain an estimate of $G_F$. Compare this to the current value of the Fermi coupling constant from the Particle Data Group [4]

$$G_F = (1.1663787 \pm 0.0000006)\times 10^{-5}\ \mathrm{GeV}^{-2}.$$

## References


[1] Schwartz, M.D., *Quantum Field Theory and the Standard Model*, Cambridge, 2017.

[2] Peskin, M. and Shroeder, D., *An Introduction to Quantum Field Theory*, CRC Press, 2019.

[3] Tong, D., *Quantum Field Theory*, Part III Maths Tripos Lecture Notes.

[4] Zyla, P.A. *et al.* (Particle Data Group), PTEP (2020) 083C01.