# Quantum field theory I : Physics 721

## Autumn / Fall 2021

## Introduction

### Outline

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* Quick Quiz 0

## The plan

This course is structured a little differently than the textbook(s) - and from last year. Quantum field theory is a deep, rich subject that builds on almost all aspects of your physics education. There are therefore nearly as many approaches to presenting the material as there are textbooks. And there are a lot of textbooks.

except maybe labs?

The outline of this course, which is subject to change, of course, is

0. Review
1. Particles and groups

We will spend a lot of time on classical things here - much longer than the textbooks

2. Quantising particles and fields
3. Relating observables to fields
4. Examples of theories

There is a more expansive list in the syllabus

After this course, you will be able to:

- explain how to "embed" particles into fields
- categorise quantities according to their spin and behaviour under the Lorentz group
- relate scattering observables to quantities calculable in QFT
- derive Feynman rules from known and previously-unseen Lagrangians
- calculate leading-order contributions to processes in simple gauge and nongauge theories.

The material in this course is drawn largely from Schwartz, but <u>considerably</u> rearranged, and interspersed with aspects of Peskin and Schroeder and Tong's lecture notes.

Basic recipe is

quantum mechanics + special relativity = quantum field theory

but in practice we use

classical field theory + quantum mechanics = quantum field theory

and one way to think of this is

$$\begin{tikzcd}
\text{classical mechanics} \arrow[r, "\text{quantisation}"] \arrow[d, "\text{fieldification}"] & \text{quantum mechanics} \arrow[d, "\text{``second fieldification''}"] \\
\text{classical field theory} \arrow[r, "\text{``second quantisation''}"'] & \text{quantum field theory}
\end{tikzcd}$$

To achieve these moves, we have to cover a lot of topics, but the key themes that will pop up through the course are

see the syllabus for a detailed list of topics

this is physics, after all

- simple harmonic oscillators
- symmetries
  - and their mathematical representation through groups
- scattering
  - and relating scattering observables to quantities calculable in quantum theory

## Why QFT?

1. Preserves locality ← contrast w/ e.g. Newtonian gravity

2. Successfully combines quantum mechanics and special relativity, which together lead to nonconservation of particle number

   ↖ Dirac equation tells us negative energy solutions exist ⇒ one particle theory is inconsistent ⇒ require new formulation in the relativistic regime

3. Explains why all fundamental particles of a given type are really, truly identical → they are all excitations of the same underlying field

   ↑ basic degrees of freedom of our Universe are "operator-valued functions of spacetime", but also "unitary representations of the Poincaré group"

# Units and scales

Three relevant fundamental units

- length, time, and mass
- better are $\hbar, c, G$ — linked to fundamental theories [QM, SR, and GR]

  unit of action, unit of velocity

- use natural units $\hbar = c = 1$

  $\Rightarrow$ everything expressed in mass (energy) units ← usually eV

  $[\text{length}] = [\text{time}] = [\text{mass}]^{-1}$

Note "Planck units" are $\hbar = c = G = k_e = k_B = 1$

Useful particle physics conversion $1 \simeq 197$ MeV fm

$\Delta E \geq \frac{\hbar c}{L}$ and $\Delta E \geq 2mc^2$

Two important scales

1. Compton wavelength $\lambda_c = \frac{\hbar}{mc}$

   Distance at which particle-antiparticle pairs become important, for a particle of mass $m$.

   concept of a pointlike particle breaks down

2. de Broglie wavelength $\lambda_B = \frac{h}{|\vec{p}|}$

   Distance at which particles behave like waves.

Note $\lambda_c < \lambda_B$.

Planck scale $= \sqrt{\frac{\hbar c}{G}} \sim 10^{19}$ GeV

$n, p \sim 1$ GeV

$\Lambda_{cosmo} \sim 10^{-3}$ eV

Observable Universe $\sim 10^{-33}$ eV

# Part 0: "Review"

## Learning outcomes

You will be able to:

- write down Lorentz transformations explicitly
- define and manipulate Einstein summation convention
- express Maxwell's equations in four-vector notation
- define and apply raising, lowering and number operators for simple harmonic oscillators
- define the Lagrangian and Hamiltonian
- derive equations of motion from the Lagrangian and Hamiltonian
- apply classical mechanics methods to fields
- define Noether's theorem
- apply Noether's theorem to spacetime and global symmetries

# Special relativity

[Schwarz 2.1]

Relativistic systems have Lorentz symmetry – their physics is unchanged under rotations and boosts, the elements of the (proper, orthochronous) Lorentz group.

Symmetries and groups are going to be important in this course, so we're going to start by revising a group you already know, even if you haven't seen it presented in this way before – the group of 3D rotations (i.e. $SO(3)$)

Boosts are basically just a generalised rotation.

S = special
O = orthogonal
3 = 3

## Rotations

We can represent a rotation around the z axis as

$$x_i \rightarrow x_i' = \sum_j R_{ij} x_j \qquad R_{ij} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The rotation matrix satisfies

$R^T R = \mathbb{1}$ ← orthogonal ↑ try it!

$\det R = 1$ ← special

Q: What is $x'$ if $x = (x, y, z)$ ?

and preserves the norm of a 3-vector (or inner or scalar product)

$$|\bar{x}| = \sqrt{\sum_i x_i x_i}$$

This provides a way to define the rotation group – the set of all transformations that leave the norm invariant

Rotations are a subgroup of the Lorentz group, which preserve the norm of four-vectors

<u>Four-vectors</u>

Vectors $x^\mu = (x^0, x^1, x^2, x^3)$ ← tangent vector, contravariant vector

Co-vectors $x_\mu = (x_0, x_1, x_2, x_3)$ ← one form

↑ Lorentz index

↑ related via the <u>metric</u> $x^\mu = g^{\mu\nu} x_\nu$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Some comments:

- "0" entries are implicit
- $\eta^{\mu\nu}$ is used for the Minkowski when considering curved spacetime with general metric $g^{\mu\nu}$ ↖ we don't need this!
- we're using the Einstein summation convention
  - $x^\alpha y_\alpha = \sum_{\alpha=0}^{3} x^\alpha y_\alpha$
  - repeated indices are summed over
  - repeated indices must come in upstairs-downstairs pairs
  - free indices must match in valid equations

Inner product is invariant under Lorentz transformations

$$x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (\bar{x})^2$$

↑ 3-vector!

Our rotation matrix now takes the form

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\vartheta & \sin\vartheta & \\ & -\sin\vartheta & \cos\vartheta & \\ & & & 1 \end{pmatrix}$$

and acts on four-vectors as

$$V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu$$

Boosts are the same, but mix up space and time.

e.g. $\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\beta & & & \sinh\beta \\ & 1 & & \\ & & 1 & \\ \sinh\beta & & & \cosh\beta \end{pmatrix}$ is a boost in the z direction with rapidity

$$\beta = \operatorname{arccosh} \frac{1}{\sqrt{1-v^2}}$$

Some useful four-vectors:

- spacetime $\quad x^\mu = (x^0, x^1, x^2, x^3)$
- momentum $\quad p^\mu = (E, \bar{p}) = (E, p^1, p^2, p^3)$

  $$p^2 = p^\mu p_\mu = E^2 - \bar{p}^2 = m^2$$

- velocity $\quad p^\mu = m_0 u^\mu \qquad u^2 = 1$
- derivatives $\quad \partial^\mu = \frac{\partial}{\partial x_\mu} \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}$ ← Q: what is $\partial^\mu x_\nu$
- d'Alembertian $\quad \Box = \nabla^2 = \partial_\mu^2 = \partial_\mu \partial^\mu$

  ↑ NOT $\bar{\nabla}^2$ !

# Maxwell's equations

in differential form

In vacuum, Maxwell's equations are

$\nabla \cdot \bar{E} = 0$ ← $= 4\pi\rho$ in presence of sources

↑ Gaussian units ü

$\nabla \cdot \bar{B} = 0$ } Gauss' law

$\nabla \times \bar{E} = -\dfrac{\partial \bar{B}}{\partial t}$ Faraday's law

$\nabla \times \bar{B} = \dfrac{\partial \bar{E}}{\partial t}$ ← $+ 4\pi \bar{J}$ in presence of sources Ampère's law

By introducing the four-vector

$A^\mu = (\phi, \bar{A})$ with $\bar{E} = -\bar{\nabla}\phi - \dfrac{\partial \bar{A}}{\partial t}$ and $\bar{B} = \bar{\nabla} \times \bar{A}$

and the (antisymmetric) tensor, the field strength

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

strictly, only the upper component version is a tensor, I am being a bit sloppy here.

we can rewrite these Maxwell equations as

$\partial_\mu F^{\mu\nu} = 0$ ← this is the equation of motion (as we'll see later)

$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ ← this is the Bianchi identity, satisfied by any antisymmetric tensor

The components of the field strength are

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix}$$

Q: What are the components of $F_{\mu\nu}$?