Part 1: "Particles and groups"

Learning outcomes - the Lorentz and Poincaré groups

You will be able to:

- define a group
- apply this definition to the Lorentz group
- understand the role of generators in Lie algebras and groups
- define the Poincaré group
- relate particle properties to irreducible unitary representations of the Poincaré group
- define and identify the transformation properties of fields under the Lorentz group
- characterise irreducible representations of the Lorentz group.

Particles and group theory

Everything we've done so far has largely been review (even if some of it has been expressed in a form that hasn't been emphasised in earlier courses). We've discussed simple harmonic oscillators, Lorentz transformations and four vectors, Maxwell's equations, and classical field theory and Noether's theorem. Clearly there are a lot of ingredients in this course...

Let's now turn to the things we're actually studying in this course – particles! To understand particles in the context of QFT, we first have to do a little group theory

Group theory

Schwarz 10.1

Tong 4.0 and P+S 3.1

A group is a set of elements and a group operation

$\{g_i\}$ "$\cdot$" or "$\times$" or "$+$"

These must satisfy

- associativity $\quad (g_i \cdot g_j) \cdot g_k = g_i \cdot (g_j \cdot g_k)$
- closure $\quad g_i \cdot g_j \in \{g_k\} \quad \forall i, j$
- identity $\quad \mathbb{1} \cdot g_i = g_i \quad \forall i$
- inverses $\quad g_i^{-1} \cdot g_i = \mathbb{1} \quad \forall i$

A representation is a particular instantiation of the group, on a set of operators that act on a vector space.

Basically, you can think of a representation as a set of matrices that act on a vector space. Roughly. Beware, though, that physicists often refer to the vector space, or the vectors in it, as the (or a) representation.

A <u>faithful representation</u> is one in which each group element gets its own matrix. The <u>trivial representation</u> $r: g_i \to \mathbb{1}$, is not one of these faithful representations.

We will study various groups in this course, such as the Lorentz group, the Poincaré group and the U(1) group of electromagnetism. Let's start with our most familiar - the Lorentz group.

Q: Persuade yourself that the set of all boosts and rotations <u>is</u> a group.

<u>Lorentz group</u>

The Lorentz group is the set of all transformations that leave the four-vector norm invariant. This can also be represented (pun intended) as the set of transformations that leave the metric invariant $\Lambda^T g \Lambda = g$.

This is actually an equation for the four-vector representation of the Lorentz group!

e.g.

$$\Lambda = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \quad \text{rotation}$$

$$\Lambda = \begin{pmatrix} \cosh\beta & & & \sinh\beta \\ & 1 & & \\ & & 1 & \\ \sinh\beta & & & \cosh\beta \end{pmatrix} \quad \text{boost}$$

act on four-vectors as $x^\mu = \Lambda^\mu{}_\nu x^\nu$

We want to move away from a specific representation of the group, so to do that we're going to consider the group generators. You can think of these as a set of "basis objects" that together "span" the group.

In other words, any group element can be "built" up by suitable combinations of generators.

More specifically, we will start with infinitesimal transformations

e.g.

$$\Lambda \simeq \begin{pmatrix} 1 & & & \\ & 1 & \theta & \\ & -\theta & 1 & \\ & & & 1 \end{pmatrix}$$

$$\Lambda \simeq \begin{pmatrix} 1 & & & \beta \\ & 1 & & \\ & & 1 & \\ \beta & & & 1 \end{pmatrix}$$

← This is allowed, because we're thinking about the Lorentz group, which is continuous

The basic idea of generators is to generalise this idea of infinitesimal transformations (any group element can be composed of infinitesimal boosts and translations) in the neighborhood of the identity element.

More concretely, we have

$$x^0 \to x'^0 = x^0 + \overbrace{\beta^i x^i}^{\delta x^0}$$

$$x^i \to x = \underbrace{+\beta^i x^0 - \epsilon^{ijk}\theta^j x^k}_{\delta x^i}$$

Q: convince yourself this works!

This can be rewritten as four-vectors multiplied by a set of basis matrices (the generators in the vector representation)

Aside

The object ϵ_{ijk} is not a tensor – it is a tensor density (or pseudotensor) in flat spacetime.

of weight 1

$$\tilde{\epsilon}_{\mu_1' \cdots \mu_n'} = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

In particular it changes sign under improper Lorentz transformations

det = −1

We can define a true tensor in curved spacetime through

$$\epsilon_{\mu_1 \mu_2 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n}.$$

square root of the determinant of the metric

In curved spacetime $\tilde{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n}$ becomes the Levi-Civita symbol, which is defined to be $\tilde{\epsilon}_{12 \cdots n} = +1$ (and so on) in any coordinate system.

The commonly defined $\tilde{\epsilon}^{\mu_1 \mu_2 \cdots \mu_n}$, whose components are numerically equal to those of $\tilde{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n}$, is a tensor density of weight −1

$$\epsilon^{\mu_1 \mu_2 \cdots \mu_n} = \frac{\mathrm{sgn}(g)}{\sqrt{|g|}} \tilde{\epsilon}^{\mu_1 \mu_2 \cdots \mu_n}$$

Note: ϵ can only be defined as a tensor on orientable manifolds
can consistently choose "clockwise" loops – or surface normal vector

But what is an improper Lorentz transformation?

To answer this, we need a little more group theory.

The Lorentz group is $O(1,3)$, which preserves $ds^2 = dt^2 - d\vec{x}^2$.

- noncompact – can't "hold it in your hands" (~ finite and bounded)
  SU(2) is compact
- not connected – not "joined up"
- 6-dimensional – = 6 generators
- non-Abelian Lie group – generators don't commute, Lie groups are also differentiable manifolds

and have smooth operations

The Lorentz group has four (separate) connected bits, which are labelled by their properties under parity and time reversal - either the elements do or do not change sign (there is no try) under these operations.

↑ not subgroups Q - why not?

or "restricted Lorentz group"

The proper orthochronous Lorentz group, $SO^+(1,3)$, contains the identity and all transformations have determinant equal to $+1$ (= "proper") and preserve the direction of time (= "orthochronous").

This connected piece contains rotations and boosts as we usually think of them, and the they can be connected continuously to the identity.

Every element in $O(1,3)$ can be written as the semi-direct product $O(1,3) \simeq SO^+(1,3) \otimes \{\mathbb{1}, P, T, PT\}$

Thus ϵ_{ijk} can be thought of as a "tensor" in $SO^+(1,3)$, but is not really a tensor, because it changes sign under parity transformations.

N.B. $SL(2,\mathbb{C})$ is the universal (double) cover of $SO^+(1,3)$, i.e. the group obtained by exponentiating the isomorphic Lie algebra $sl(2,c) \simeq su(2) \oplus su(2)$. By definition the universal cover is simply connected

By introducing

$$J_1 = i\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{pmatrix} \qquad J_2 = i\begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0 \end{pmatrix} \qquad J_3 = i\begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$K_1 = i\begin{pmatrix} 0&-1&0&0\\ -1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \qquad K_2 = i\begin{pmatrix} 0&0&-1&0\\ 0&0&0&0\\ -1&0&0&0\\ 0&0&0&0 \end{pmatrix} \qquad K_3 = i\begin{pmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ -1&0&0&0 \end{pmatrix}$$

we can write

<u>not</u> a tensor!

generators

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu = \left(\delta^\mu{}_\nu + \underbrace{i\sum_{i=1}^{3}\left[\theta_i (J_i)^\mu{}_\nu + \beta_i (K_i)^\mu{}_\nu\right]\Big) x^\nu}_{\delta x^\mu}$$

We can generate (pun intended) arbitrary group elements by repeated application of this – that is, we exponentiate the generators.

$$\Lambda^\mu{}_\nu = \exp\left(i\theta_i (J_i)^\mu{}_\nu + i\beta_i (K_i)^\mu{}_\nu\right)$$

Q: why?

In this case we are still in the vector representation. But this holds for all representations!

$$\boxed{\Lambda = \exp\left(i\sum_{i=1}^{3}(\theta_i J_i + \beta_i K_i)\right)}$$

Notes and comments:

- this all works because the Lorentz group is a Lie group - a smooth differentiable manifold.
- the generators form the Lie algebra
- Lie groups are continuous (by definition)
- the full Lorentz group includes discrete symmetries and is therefore not simply connected and not actually a Lie group. Only the proper orthochronous Lorentz group $SO^+(1,3)$ can be reached continuously from the identity. We will discuss this in more detail later, but see the Aside on page 19 of these notes.
- Lie groups are everywhere in particle physics - $SU(2)$, $SO(3)$, $U(1)$, $SU(3)$, ...

- Tong uses rather different notation in his chapter 4. He expresses the general relation as

$$\Lambda = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma} M^{\rho\sigma}\right)$$

Tong eq. 4.12

parameters of transformation, ie six numbers corresponding to three angles and three rapidities

generators, ie six 4x4 matrices J_i, K_i

in Schwarz these are the $V^{\mu\nu}$ of equation 10.20

- Peskin and Schroeder use a different notation again. In the vector representation, they write

$$V^\alpha \to \left(\delta^\alpha{}_\beta - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha{}_\beta\right)V^\beta \qquad \text{P+S eq. 3.19}$$

This leads us to one of the key facts of QFT

> Keeping track of indices and of different conventions and notations are the two hardest aspects of quantum field theory.

Recall that there is a group operation $g_i \cdot g_j = g_k$. The group structure that this operation satisfies translates into the <u>Lie algebra</u>, which defines the operation at the level of the generators, through the <u>Lie bracket</u> ← which you know as a commutator

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad \leftarrow \text{the } J_i \text{ generate the 3D subgroup } SO(3)$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

The Lie algebra corresponding to the Lorentz group is the <u>Lorentz algebra</u>, denoted $so(1,3)$. The group is $SO(1,3)$.

lowercase! uppercase!

If we define

$$V^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

each entry is <u>itself</u> a 4x4 matrix!

then the Lorentz algebra is

$$\boxed{[V^{\mu\nu}, V^{\rho\sigma}] = i\left(g^{\nu\rho}V^{\mu\sigma} - g^{\mu\rho}V^{\nu\sigma} - g^{\nu\sigma}V^{\mu\rho} + g^{\mu\sigma}V^{\nu\rho}\right)}$$

<u>Fields and the Lorentz group</u>

Our discussion has so far been about the vector representation of the Lorentz group, even though we have written down more general expressions, valid for any representation. Basically we've had in mind how spacetime itself (more specifically, the position in spacetime) changes under a Lorentz transformation. But what about functions of spacetime, or fields?

At this point, it is helpful to think of the Lorentz transformations as coordinate transformations (after all, a boost is nothing other than moving to a reference frame with a different relative velocity) and remember that, say, vectors themselves don't change, only their components.

in a specific reference frame

Let's make this concrete : consider a rotation

$$x \to x' = Rx$$

and how it acts on two types of (classical) field

← locally, in Williamsburg

- temperature field
- Earth's magnetic field

↑ "geomagnetic field"

↖ locally, on the Earth's surface, in Williamsburg

Clearly the temperature field does not change under rotations – the temperature outside does not drop 10°C if you turn 90° to your right.

more precisely, its components in your reference frame change

Geomagnetic field does change !

- suppose you define a z-axis to point in the direction you're currently facing, parallel to the Earth's surface.
- now face roughly 10° west of north and the geomagnetic field will be aligned with your z axis
- turn 90° to your right and hey, presto ! the magnetic field no longer points along your z axis.

⇒ temperature field = a <u>scalar field</u>

geomagnetic field = a <u>vector field</u>

This example leads us to two keys to getting to grips with the relationship between particles, fields and Lorentz grp.

$$\boxed{\text{1. Distinguish the effect of Lorentz transforms on spacetime coordinates and on functions of those coordinates, that is, on fields.}}$$

In QFT we are generally not interested in totally random functions of spacetime, we are interested in ones that have specific transformation properties under Lorentz transformations

$$\boxed{\text{2. Fields are functions of spacetime that transform as specific representations of the Lorentz group}}$$

Not quite true in all generality, but in this course we can always write more general things in terms of fields that do satisfy this.

* Quick quiz 1

The Poincaré group

$\boxed{\text{Schwartz 8.1}}$

The Poincaré group is the Lorentz group plus all spacetime translations

$x \to x' = x + a$ for a any constant four-vector

Q: We need six generators for the Lorentz group. How many do we need for the Poincaré group?