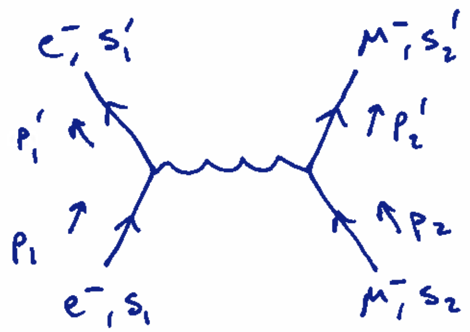
1. Diagram is



2. $$i\mathcal{M} = \bar{u}^{s_1'}(\bar{p}_1')(-ig\gamma^\mu)u^{s_1}(\bar{p}_1) \cdot \frac{-ig_{\mu\nu}}{(p_1-p_1')^2+i\epsilon} \cdot \bar{u}^{s_2'}(\bar{p}_2')(-ie\gamma^\nu)u^{s_2}(\bar{p}_2)$$

3. $$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

$$= \frac{1}{4}\frac{g^4}{((p_1-p_1')^2+i\epsilon)^2} \mathrm{Tr}\left[(\not{p}_1' + m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right]$$
$$\times \mathrm{Tr}\left[(\not{p}_2' + m_\mu)\gamma_\mu(\not{p}_2+m_\mu)\gamma_\nu\right]$$

Recall that our expression for $e^+e^- \to \mu^+\mu^-$ was

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\frac{g^4}{(p+p')^4} \mathrm{Tr}\left[(\not{p}+m_e)\gamma^\mu(\not{p}'-m_e)\gamma^\nu\right]$$
$$\times \mathrm{Tr}\left[(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\right]$$

So if we make the substitutions

$p \to p_1$
$p' \to -p_1'$
$k \to p_2'$
$k' \to -p_2$

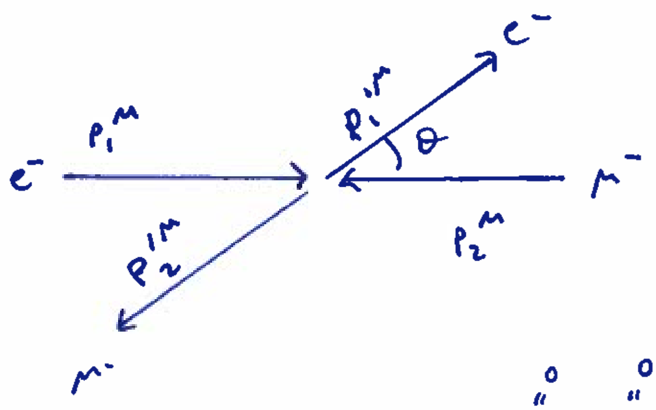
and the fact that
$$\mathrm{Tr}\left[(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\right] = \mathrm{Tr}\left[(\not{k}+m_\mu)\gamma_\nu(\not{k}'-m_\mu)\gamma_\mu\right]$$

Then we can use the old results!

So we immediately get

$$\overline{|M|^2} = \frac{8g^4}{(p_1-p_1')^4}\Big[(p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2') - m_e^2(p_2\cdot p_2') - m_\mu^2\, p_1\cdot p_1' + 2m_e^2 m_\mu^2\Big]$$

[4.] We work in the CM frame, with $\frac{m_e}{E} \ll 1$



$$p_1^\mu = (|\bar{k}|, |\bar{k}|\hat{z})$$
$$p_2^\mu = (E, -|\bar{k}|\hat{z})$$
$$p_1'^\mu = (|\bar{k}|, \bar{k})$$
$$p_2'^\mu = (E, -\bar{k})$$

$\nwarrow$ $E^2 = |\bar{k}|^2 + m_\mu^2$
$\bar{k}\cdot\hat{z} = |\bar{k}|\cos\theta$
$E + k = E_{cm}$

Then
$$(p_1-p_1')^2 = \overset{0}{\overset{\shortparallel}{p_1^2}} + \overset{0}{\overset{\shortparallel}{p_1'^2}} - 2p_1\cdot p_1'$$
$$= -2(|\bar{k}|^2 - |\bar{k}|^2\cos\theta)$$
$$p_1\cdot p_1' = |\bar{k}|^2(1-\cos\theta)$$
$$p_1\cdot p_2' = E|\bar{k}| + |\bar{k}|^2\cos\theta = |\bar{k}|(E + |\bar{k}|\cos\theta) = p_1'\cdot p_2$$
$$p_1\cdot p_2 = E|\bar{k}| + |\bar{k}|^2 = |\bar{k}|(E+|\bar{k}|) = p_1'\cdot p_2'$$

So we have

$$\overline{|M|^2} = \frac{8g^4}{4|\bar{k}|^4(1-\cos\theta)^2}\Big[|\bar{k}|^2(E+|\bar{k}|\cos\theta)^2 + |\bar{k}|^2(E+|\bar{k}|)^2 - m_\mu^2|\bar{k}|^2(1-\cos\theta)\Big]$$

$$= 2g^4\frac{1}{|\bar{k}|^2(1-\cos\theta)^2}\Big[E^2 + 2E|\bar{k}|\cos\theta + |\bar{k}|^2\cos^2\theta + E^2 + 2E|\bar{k}| + |\bar{k}|^2 - m_\mu^2(1-\cos\theta)\Big]$$

$$= 2g^4\frac{1}{|\bar{k}|^2(1-\cos\theta)^2}\Big[2E^2 + 2E|\bar{k}|(1+\cos\theta) + |\bar{k}|^2(\cos^2\theta+1) + m_\mu^2(\cos\theta-1)\Big]$$

$\boxed{5.}$ Our expression for the cross-section is pretty simple when one particle is massless

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\bar{p}_1|}{(2\pi)^2 4E_{cm}} |M|^2 \rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|M|^2}{64\pi^2 \underset{E_{cm}}{\underbrace{(E+k)}}^2}$$

Thus

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\alpha^2}{2k^2 E_{cm}^2 (1-\cos\theta)^2}\left(E_{cm}^2 + (E+k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right)$$

N.B.

$$\frac{|\bar{p}_1|}{2E_A 2E_B |v_A - v_B| (2\pi)^2 4E_{cm}} = \frac{|\bar{E}|}{16(2\pi)^2 E|\bar{k}|\left|\frac{\bar{k}}{\bar{k}} + \frac{\bar{k}}{E}\right|(E+k)} = \frac{1}{16(2\pi)^2 E\left|1+\frac{k}{E}\right|(E+k)} = \frac{1}{64\pi^2 E_{cm}^2}$$

And if $E \gg m_\mu$ we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\alpha^2}{2E_{cm}^2(1-\cos\theta)^2}\left(4 + (1+\cos\theta)^2\right)$$

For small angles

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} \propto \frac{1}{\theta^4}$$

$\Rightarrow$ this diverges for back-to-back scattering – this is a feature of Coulomb scattering and arises from the photon going "onshell"

The trick we used to replace the momenta, rather than doing all the trace algebra again, is our first example of <u>crossing symmetry</u>:

amplitude for process with <u>particle</u> of momentum p in <u>initial state</u>	=	amplitude for process with <u>antiparticle</u> of momentum $-p$ in <u>final state</u>

↑ amplitude = S-matrix element

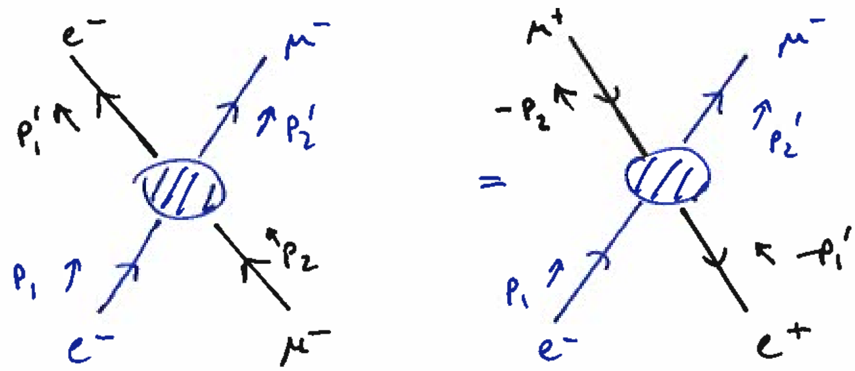
↑ strictly this means "analytic continuation of"

Schematically, we can write this as

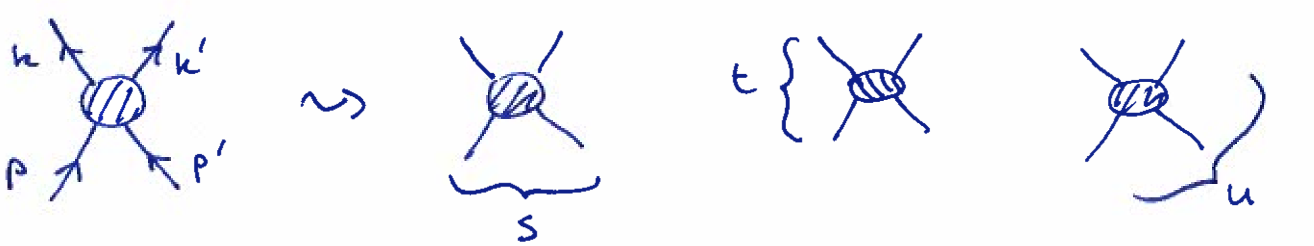
$$i\mathcal{M}(\phi(p)\, X \to X') = i\mathcal{M}(X \to \bar{\phi}(-p)\, X')$$

We do, however, have to be careful when the external particles have spin, but we don't need the details here. See P+S p. 155 for the details.

In our case we crossed



Crossing symmetry is particularly manifest with the <u>Mandelstam variables</u> that we have seen before.



$s = (p+p')^2$ $\quad$ $t = (k-p)^2$ $\quad$ $u = (k'-p)^2$

$[\, = (k+k')^2\,]$ $\quad$ $[\, = (k'-p')^2\,]$ $\quad$ $[\, = (k-p')^2\,]$

For equal masses, we have

For physical scattering $s > 0$, $t < 0$

$$s = (p+p')^2 = (2E)^2 = E_{cm}^2$$

$$t = (k-p)^2 = -2|\bar{p}|^2(1-\cos\theta)$$

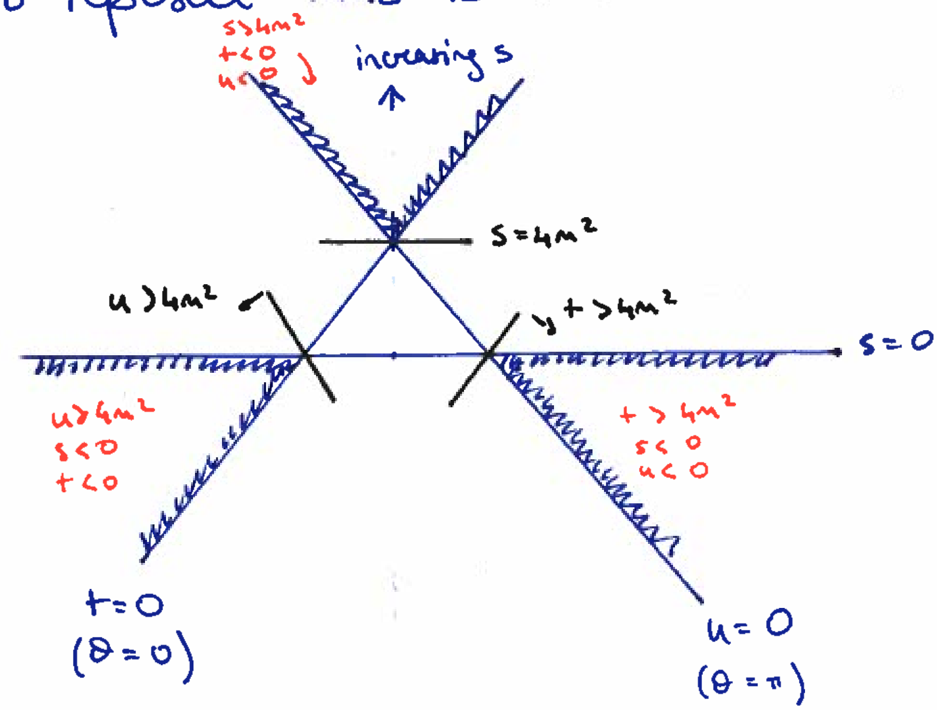
$$u = (k'-p)^2 = -2|\bar{p}|^2(1+\cos\theta)$$

There are only two independent variables, since

$$s + t + u = 4m^2$$

We have traded (E_{cm}, θ) for (s, t), which are Lorentz invariants.

One way to represent this is the Mandelstam plane



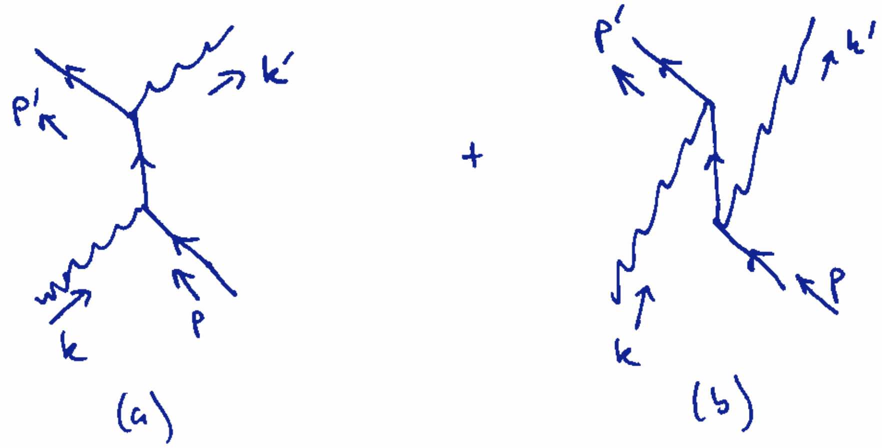
Our scattering examples in QED have focussed on processes with external fermions, but one of the processes we wrote down at the beginning of this section was Compton scattering, which involves external photons. So let's see what complications they bring.

Schwartz 13.5
P+S 5.5

Compton scattering

Recall this is $e^{\pm}\gamma \to e^{\pm}\gamma$

$\boxed{1.}$



$\boxed{2.}$ $i\mathcal{M} = i\mathcal{M}_a + i\mathcal{M}_b$

where

$$i\mathcal{M}_a = (-ig)^2\, \epsilon^*_\mu(\bar{k}', \lambda')\, \epsilon_\nu(\bar{k}, \lambda)\, \bar{u}^{s'}(\bar{p}')\gamma^\mu \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i\epsilon}\gamma^\nu u^s(\bar{p})$$

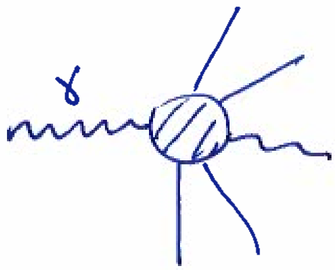
$$i\mathcal{M}_b = (-ig)^2\, \epsilon^*_\mu(\bar{k}', \lambda')\, \epsilon_\nu(\bar{k}, \lambda)\, \bar{u}^{s'}(\bar{p}')\gamma^\nu \frac{i(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2 + i\epsilon}\gamma^\mu u^s(\bar{p})$$

It's clear that when we calculate $|\mathcal{M}|^2$ we will need to deal with sums and squares over polarisation vectors ...

Before we figure out what to do with these, we first consider a seeming tangent – the "Ward Identity"

like the Bourne Identity but more physics-y and exciting

Consider a diagram with an external photon



this must have a part where the photon couples to a fermion

1. If fermion is external $\Rightarrow \propto \epsilon_\mu(\bar{k},\lambda)\,\bar{u}(\bar{p}')\,\gamma^\mu u(\bar{p})$

2. If fermion is internal $\Rightarrow \propto \epsilon_\mu(\bar{k},\lambda)\,(\not{p}'+m)\,\gamma^\mu(\not{p}+m)$

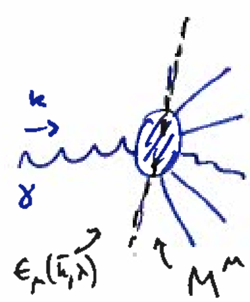
$= \sum_s u^s(\bar{p})\,\bar{u}^s(\bar{p})$

still get $\epsilon_\mu(\bar{k},\lambda)\,\bar{u}'\gamma^\mu u$

Now momentum conservation means $k^\mu = p'^\mu - p^\mu$

So replacing ϵ_μ by k_μ gives

$$
\begin{aligned}
k_\mu \bar{u}(\bar{p}')\gamma^\mu u(\bar{p}) &= (p'-p)_\mu \bar{u}(\bar{p}')\gamma^\mu u(\bar{p}) \\
&= \bar{u}(\bar{p}')(\not{p}'-\not{p})u(\bar{p}) \\
&= 0 \quad !
\end{aligned}
$$



This is, in fact, true in general

$$\boxed{\text{If } \mathcal{M} = \epsilon_\mu(\bar{k},\lambda)\,\mathcal{M}^\mu \text{ then } k_\mu \mathcal{M}^\mu = 0}$$

the Ward Identity

field theoretic expression of current conservation implied by gauge invariance

I think $\partial_\mu j^\mu = 0$ in FT

The Ward identity helps us understand $\sum_\lambda \epsilon_\mu(\bar{k},\lambda)\,\epsilon^*_\nu(\bar{k},\lambda)$

For external (on-shell) photons, there are only two polarisations

eg, choose $k^\mu = (k, 0, 0, k)$ $\quad k = |\bar{k}|$

recall this appeared in our earlier scattering example

so then $\epsilon^\mu(\bar{k}, \lambda = \pm 1) = \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$

We have

$$\sum_\lambda \epsilon^\mu(\bar{k},\lambda)\,\epsilon^{\nu *}(\bar{k},\lambda)\, M_\mu M_\nu^* = |M^1|^2 + |M^2|^2$$

But recall

$$k_\mu M^\mu = 0 \quad \Rightarrow \quad k^0 M^0 - k^3 M^3 = 0 \quad \Rightarrow k M^0 = k M^3 \quad \Rightarrow M^3 = M^0$$

Then we can add this to the right hand side

$$\sum_\lambda \epsilon^\mu(\bar{k},\lambda)\,\epsilon^{\nu *}(\bar{k},\lambda)\, M_\mu M_\nu^* = |M^1|^2 + |M^2|^2 + |M^3|^2 - |M^0|^2$$

$$= -g^{\mu\nu} M_\mu M_\nu^*$$

So we can use

$$\boxed{\sum_\lambda \epsilon^\mu(\bar{k},\lambda)\,\epsilon^{\nu *}(\bar{k},\lambda) \rightarrow -g^{\mu\nu}}$$

N.B. Provided we contract this sum with the rest of an invariant matrix element in QED, or other gauge invariant theory

Now let's see how we can use this in our expression for the Compton amplitude