3.1

Let's now take our contributions in turn

$$iM_a = \frac{-ig^2}{2p\cdot k}\, \epsilon^*_\mu(\bar{k}',\lambda')\,\epsilon_\nu(\bar{k},\lambda)\,\bar{u}^{s'}(\bar{p}')\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu u^s(\bar{p})$$

$\downarrow A^* = (A^T)^+$

$$\Rightarrow (iM_a)^* = \frac{ig^2}{2p\cdot k}\,\epsilon_\rho(\bar{k}',\lambda')\,\epsilon^*_\sigma(\bar{k},\lambda)\, u^{s+}(\bar{p})\gamma^{\sigma+}(\not{p}+\not{k}+m)^+\gamma^{\rho+}\gamma^{0+}u^{s'}(\bar{p}')$$

$$\underbrace{u^{s+}\gamma^0}_{\bar{u}^s}\underbrace{\gamma^0\gamma^{\sigma+}\gamma^0}_{\gamma^\sigma}\underbrace{\gamma^0(\not{p}+\not{k}+m)^+\gamma^0}_{\not{p}+\not{k}+m}\underbrace{\gamma^0\gamma^{\rho+}\gamma^0}_{\gamma^\rho}u^{s'}$$

$$= \frac{ig^2}{2p\cdot k}\,\epsilon_\rho(\bar{k}',\lambda')\,\epsilon^*_\sigma(\bar{k},\lambda)\,\bar{u}^s(\bar{p})\gamma^\sigma(\not{p}+\not{k}+m)\gamma^\rho u^{s'}(\bar{p}')$$

Then

$$\overline{|M_a|^2} = \frac{1}{4}\sum_{\substack{s,s'\\ \lambda,\lambda'}}|M_a|^2$$

$-g_{\nu\sigma}$ $\qquad$ $-g_{\rho\mu}$

$$= \frac{1}{4}\left(\frac{g^2}{2p\cdot k}\right)^2\left(\sum_\lambda \epsilon_\nu\epsilon^*_\sigma\right)\left(\sum_{\lambda'}\epsilon'_\rho\epsilon'^*_\mu\right)$$

$$\times \mathrm{Tr}\Big[\underbrace{\Big(\sum_s u^s\bar{u}^s\Big)}_{\not{p}+m}\gamma^\sigma(\not{p}+\not{k}+m)\gamma^\rho\underbrace{\Big(\sum_{s'}u^{s'}\bar{u}^{s'}\Big)}_{\not{p}'+m}\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu$$

$$= \frac{g^4}{16(p\cdot k)^2}\mathrm{Tr}\left[(\not{p}+m)\gamma^\nu(\not{p}+\not{k}+m)\gamma^\mu(\not{p}'+m)\gamma_\mu(\not{p}+\not{k}+m)\gamma_\nu\right]$$

To simplify this trace we note

$$\gamma^\mu(\not{p}'+m)\gamma_\mu = -2(\not{p}'-2m)$$

$$\gamma_\nu(\not{p}+m)\gamma^\nu = -2(\not{p}-2m) \quad \leftarrow \text{used cyclic property of trace to get LHS}$$

$$\Rightarrow \mathrm{Tr}[\ ] = 4\,\mathrm{Tr}[(\not{p}-2m)(\underbrace{\not{p}+\not{k}}_{\not{a}}+m)(\not{p}'-2m)(\not{p}+\not{k}+m)]$$

$$= 4\,\mathrm{Tr}[(\not{p}\not{a}-2m\not{a}+m\not{p}-2m^2)(\not{p}'\not{a}-2m\not{a}+m\not{p}'-2m^2)]$$

$$= 4\,\mathrm{Tr}[\not{p}\not{a}\not{p}'\not{a}-2m^2\not{p}'\not{a}+4m^2a^2-2m^2\not{p}\not{a}-2m^2\not{a}\not{p}'+m^2\not{p}\not{p}'-2m\not{p}\not{a}+4m^4]$$

$$= 4\left\{4\left(a\cdot p\, a\cdot p' - p\cdot p' a^2 + a\cdot p\, a\cdot p'\right) - 8m^2 a\cdot p' + 16m^2 a^2 \right.$$
$$\left. - 8m^2 a\cdot p - 8m^2 a\cdot p' + 4m^2 p\cdot p' - 8m^2 a\cdot p + 16m^4 \right\}$$

$$= 16\left\{ 2a\cdot p\, a\cdot p' - p\cdot p' a^2 - 4m^2 a\cdot p - 4m^2 a\cdot p' + 4m^2 a^2 + m^2 p\cdot p' + 4m^4 \right\}$$

$$= 16\left\{ 2a\cdot p\, a\cdot p' - p\cdot p' a^2 + 4m^2\left[a^2 + m^2 - a\cdot p - a\cdot p'\right] + m^2 p\cdot p' \right\}$$

We can simplify these expressions by using the kinematics

$p^2 = p'^2 = m^2$

$k^2 = k'^2 = 0$ ← massless photons

$p + k = p' + k'$ ← momentum conservation

$\Rightarrow (p+k)^2 = (p'+k')^2$ or $(p-p')^2 = (k-k')^2$

$m^2 - p\cdot p' = -k\cdot k'$

To find $k\cdot k'$ we use

$p\cdot(p + k - k') = m^2 + p\cdot k - p\cdot k'$ $[= p\cdot p']$

$\Rightarrow k\cdot k' = p\cdot p' - m^2 = p\cdot k - p\cdot k'$

Then we need

$a\cdot p = (p+k)\cdot p = m^2 + p\cdot k$

$a\cdot p' = (p+k)\cdot p' = (p + k)\cdot(p + k - k') = m^2 + 2p\cdot k - k\cdot k'$

$= m^2 + p\cdot k + p\cdot k'$

↑

$a\cdot p + a\cdot p' = 2m^2 + 2p\cdot k + p\cdot k'$

$a^2 = (p+k)^2 = m^2 + 2p\cdot k$

$p\cdot p' = m^2 + p\cdot k - p\cdot k'$

Plugging these in we obtain

$$\mathrm{Tr}[\;] = 32(m^4 + m^2 p\cdot k + p\cdot k\, p\cdot k')$$

$$\Rightarrow \overline{|\mathcal{M}_a|^2} = 2g^4\left(\left(\frac{m^2}{p\cdot k}\right)^2 + \frac{m^2}{p\cdot k} + \frac{p\cdot k'}{p\cdot k}\right)$$

Now, the whole matrix element is $i\mathcal{M} = i\mathcal{M}_a + i\mathcal{M}_b$

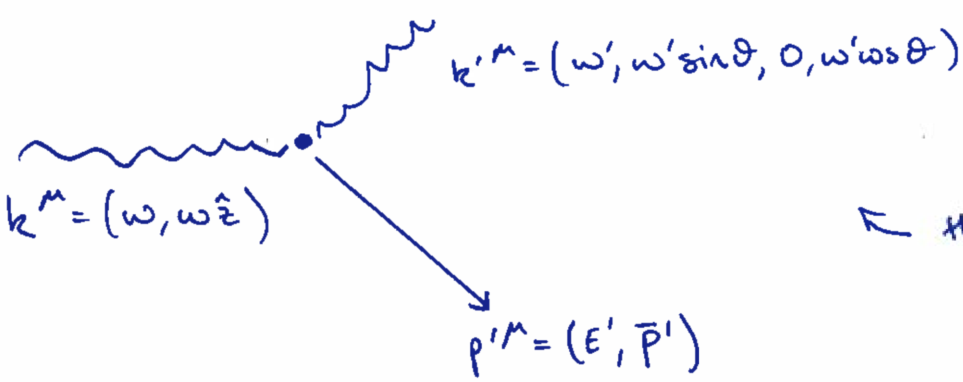
$$\Rightarrow |\mathcal{M}|^2 = (i\mathcal{M}_a + i\mathcal{M}_b)(i\mathcal{M}_a + i\mathcal{M}_b)^*$$
$$= |\mathcal{M}_a|^2 + |\mathcal{M}_b|^2 + 2\mathrm{Re}(\mathcal{M}_a\mathcal{M}_b)$$

Repeating this analysis for these second and third terms gives

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{ss'}\sum_{\lambda\lambda'}|\mathcal{M}|^2$$
$$= 2g^4\left[\frac{p\cdot k}{p\cdot k'} + \frac{p\cdot k'}{p\cdot k} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]$$

for more details, but organised slightly differently, see P+S 5.5

4. In contrast to other processes that we've studied, a good frame to study this process in is one in which the initial electron is at rest.



then $p\cdot k = m\omega$
$p\cdot k' = m\omega'$

To solve for ω', we can use brute force

First we note cons. of mom:

$$\omega + m = \omega' + E'$$
$$p'_x + \omega' \sin\theta = 0$$
$$p'_y = 0$$
$$p_z + \omega' \cos\theta = \omega$$

$$E'^2 = |\bar{p}'|^2 + m^2$$
$$= \omega'^2 \sin^2\theta + (\omega - \omega' \cos\theta)^2 + m^2$$
$$= \omega^2 + \omega'^2 - 2\omega\omega' \cos\theta + m^2$$

and

$$E'^2 = (\omega + m - \omega')^2$$
$$= \omega'^2 + (\omega + m)^2 - 2\omega'(\omega + m)$$

So equating these gives

$$-2\omega\omega' - 2\omega' m + 2\omega m = -2\omega\omega' \cos\theta$$
$$2\omega m = 2\omega'(\omega + m - \omega\cos\theta)$$
$$\Rightarrow \omega' = \frac{\omega}{1 + \frac{\omega}{m}(1 - \cos\theta)}$$

Or we can use the trick from P+S p. 162

$$m^2 = (p')^2 = (p + k - k')^2 = p^2 + 2p\cdot(k - k') - 2k\cdot k'$$
$$= m^2 + 2m(\omega - \omega') - 2\omega\omega'(1 - \cos\theta)$$
$$\Rightarrow \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1 - \cos\theta)$$

← Compton's formula for the shift in photon wavelength

$\boxed{5.}$ Finally we want to find the cross-section

$$d\sigma = \frac{1}{2E_A 2E_B |v_B - v_A|} \left(\prod_f \frac{d^3\bar{p}_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(p_A, p_B \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f p_f)$$

↑ Eq 4.79 P+S

We write this as

$$d\sigma = \frac{1}{2\omega\, 2m\, |1-0|} \int \frac{d^3\bar{k}'}{(2\pi)^3} \int \frac{d^3\bar{p}'}{(2\pi)^3} \frac{1}{2\omega'} \frac{1}{2E'} (2\pi)^3 \delta^{(3)}(\underset{0}{\bar{p}} + \underset{\omega\hat{z}}{\bar{k}} - \bar{p}' - \bar{k}')$$

$$\times (2\pi)\delta(m+\omega-\omega'-E') \qquad \Rightarrow \bar{p}' = \omega\hat{z}' - \bar{k}'$$

$$\times |M(p_A, p_B \to \{p_f\})|^2$$

$$= \frac{1}{4m\omega} \int \frac{\omega'^2 d\omega'}{(2\pi)^3} \int d\Omega\, |M|^2 (2\pi)\, \delta\left(m+\omega-\omega' - \sqrt{m^2+\omega^2-\omega'^2-2\omega\omega'\cos\theta_k}\right)$$

$$= \frac{1}{16m\omega} \frac{(2\pi)}{(2\pi)^3} \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, |M|^2 \int \frac{d\omega'\, \omega'^2}{\omega' E'} \delta(f(\omega'))$$

To evaluate the $\delta(f(\omega'))$ we use

$$\delta(f(\omega)) = \frac{1}{\left|\frac{\partial f}{\partial \omega'}\right|_{\omega'=0}} \delta(\omega' - \omega_0') \qquad \text{where } \omega_0' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)} \text{ is solution of } f(\omega') = 0$$

$$= \frac{1}{1 + \frac{\omega' - \omega\cos\theta}{E'}}$$

$$\frac{d}{d\omega'} f(\omega') = \frac{d}{d\omega'}\left(m+\omega-\omega' - \sqrt{m^2+\omega^2-\omega'^2-2\omega\omega'\cos\theta}\right)$$

$$= -1 - \frac{1}{2}\frac{2\omega' - 2\omega\cos\theta}{\sqrt{m^2+\omega^2-\omega'^2-2\omega\omega'\cos\theta}}$$

$$= -\left(1 + \frac{\omega' - \omega\cos\theta}{E'}\right)$$

So

$$d\sigma = \frac{1}{16m\omega} \frac{1}{2\pi} \int_{-1}^{1} d(\cos\theta) \frac{\omega'}{E'} \frac{1}{1+\frac{\omega'-\omega\cos\theta}{E'}}$$

$$\times 2g^4 \Bigg[\underbrace{\frac{m\omega}{m\omega'} + \frac{m\omega'}{m\omega} + 2m^2\left(\frac{1}{m\omega} - \frac{1}{m\omega'}\right) + m^4\left(\frac{1}{m\omega} - \frac{1}{m\omega'}\right)^2}_{\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta} \Bigg]$$

$$\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}$$

We end up with

$$\frac{d\sigma}{d\cos\vartheta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\vartheta\right]$$

the spin-averaged "<u>Klein-Nishina formula</u>"

Limits

- $\omega \ll m;\ \omega' \simeq \omega$ ← low energy / long wavelength limit

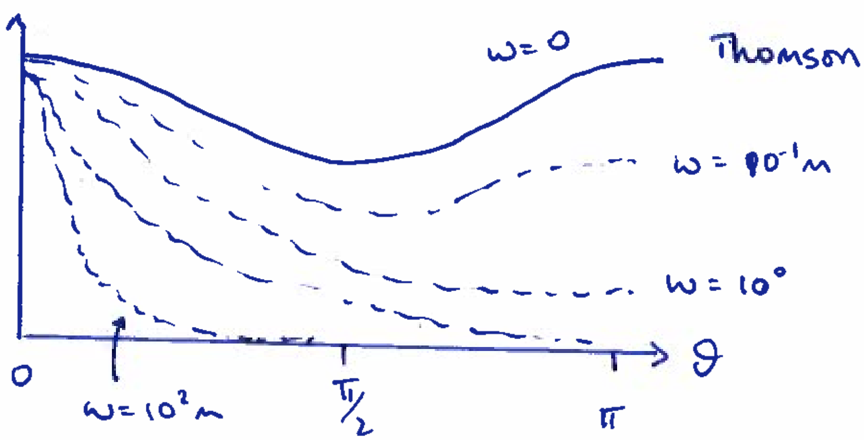
$$\frac{d\sigma}{d\cos\vartheta} \longrightarrow \frac{\pi\alpha^2}{m^2}(1+\cos^2\vartheta)$$

"<u>Thomson scattering</u>" classical photon-electron scattering cross-section

- $\omega \gg m$

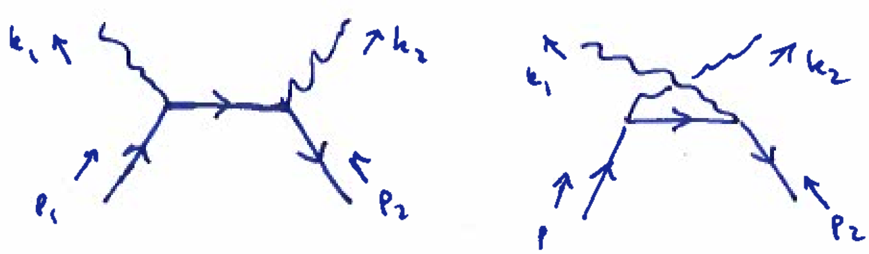
very forward peaked

See P+S p. 164-167



<u>Note</u>

We can obtain pair production of photons by crossing symmetry



See P+S p. 168-169