Particles come in many, many types, and they are all distinguished by their quantum numbers – mass, spin, orbital angular momentum, ...

Some of the quantum numbers change under Poincaré transformations, some essential ones do not. If you move your experiment 1m to the left, you'd be shocked if all your electrons turned into quarks. Thus a particle is characterised by a set of states that only mix among themselves under Poincaré transformations.

We can represent (pun intended) this as

$$|\psi_i\rangle \to |\psi_i'\rangle = P_{ij}|\psi_j\rangle$$

set of states that mix among themselves

representation of the Poincaré group

in other words, particles are objects that transform under representations of the Poincaré group.

N.B. Schwartz himself conflates the states with representation just below equation 8.1, even though he cautions about this later, on page 159.

The smallest set of states that mix among themselves transform under an <u>irreducible</u> <u>representation</u>

in other words, there is no subset of states that transform only amongst themselves

in practice, this means that the set of matrices representing this representation cannot be made block specifically matrix elements

Moreover, we want to compute quantities that are Poincaré invariant, because it makes our lives easier.

For example, suppose we want to calculate

$M = \langle \varphi_2 | \varphi_1 \rangle$ ← in QM this is the overlap of state $|\varphi_2\rangle$ on state $|\varphi_1\rangle$

Poincaré invariance means

$$M \to M' = \langle \varphi_2 | P^\dagger P | \varphi_1 \rangle = M \Rightarrow \underbrace{P^\dagger P = 1}_{\text{the definition of a unitary transformation}}$$

Putting this all together:

$$\boxed{\text{Particles transform under irreducible unitary representations of the Poincaré group}}$$

↑ This basically defines what a particle is! (once we understand how to quantise it!)

The focus on irreducible representations ("irreps") explains why some quantum numbers are fundamental – like spin – and some are not – like four-momentum.

Embedding particles into fields

Schwartz 8.2

There are no finite-dimensional unitary representations of the Poincaré group.

↑ to get around this, we will use the infinite dimensional representations labelled by momentum

Irreducible unitary representations are uniquely classified by mass m and spin j, with $m \geq 0$ and $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

- $j=0 \Rightarrow$ one state for any m
- $j>0 \Rightarrow$ $\left.\begin{array}{l}\text{two states if } m=0 \\ 2j+1 \text{ states if } m>0\end{array}\right\}$ for every value of $p^2 = m^2$

This classification of particles (i.e. labelling them by mass and spin) makes things tricky, because we want to represent particles in terms of fields with specific Lorentz transformation properties (i.e. objects with no (scalar), one (vector), two (tensor), or more Lorentz indices). Put another way, particles have $2j+1$ ← if $m>0$ degrees of freedom, but tensors have 4^n ← $n \geq 0$ degrees of freedom. This discrepancy leads to redundancy in our description.

↖ i.e. gauge symmetry

General Lorentz group representations

Schwarz 10.1.2

In contrast to the Poincaré group, the irreducible representations of the Lorentz group are characterised by two half integers.

This is because the Lie algebra of the Lorentz group, so(1,3), actually has two (commuting) subalgebras, su(2).

to see this

To see this, start with the rotation and boost generators and consider the linear combinations

$$J_i^{\pm} = \frac{1}{2}\left(J_i \pm iK_i\right) \leftarrow \text{satisfy} \quad \begin{aligned} [J_i^+, J_j^+] &= i\epsilon_{ijk} J_k^+ \\ [J_i^-, J_j^-] &= i\epsilon_{ijk} J_k^- \\ [J_i^+, J_j^-] &= 0 \end{aligned}$$

Two independent copies of the su(2) Lie algebra!

N.B.
$so(3) = sl(2,\mathbb{R})$
$= so(2,1)$
$= su(2)$

Recall that representations of su(2) are labelled by the non negative half-integer j ← basically angular momentum / spin / etc.

And for each j, there are $2j+1$ independent states, ← usually labelled "m_j" or "m"

Irreducible representations of the Lorentz group are labelled by two (non-negative) half-integers.

General Lorentz group irreducible representation:

- labelled by (j_1, j_2)
- have $(2j_1+1)(2j_2+1)$ degrees of freedom
- generate representations of $so(3)$ with spins   ← subgroup of $so(1,3)$

  $$j = j_1 + j_2,\ j_1 + j_2 - 1,\ \ldots,\ |j_1 - j_2|$$

- regular rotation generators are $\vec{J} = \vec{J}^{+} + \vec{J}^{-}$

To describe particles, we use mass and spin, but to write down Lagrangians built from fields, described by two half-integer quantum numbers. Thus we need to decompose representations of $su(2) \oplus su(2)$ into representations of $so(3)$

$so(3)$ ↑ spin algebra

$su(2) \oplus su(2)$ ↑ Lorentz algebra

$su(2) \oplus su(2)$	$(0,0)$	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(1,0)$
$so(3)$	0	$\frac{1}{2}$	$\frac{1}{2}$	$1 \oplus 0$	1

↑ Table 10.1 Schwartz

4-vector (eg A^{μ})

spin 1 or spin 0