Quantum Field Theory I: PHYS 721
Problem Set 3: Solutions

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Overview

The questions in this problem give you some practice at manipulating spinors and summarising how particles and fields relate.

Question 1 **[10]**

(a) Prove the spinor relation

$$\overline{u}^r(\vec{p})\gamma^\mu u^s(\vec{q}) = \frac{1}{2m}\overline{u}^r(\vec{p})\left[p^\mu + q^\mu + i\sigma^{\mu\nu}(p_\nu - q_\nu)\right]u^s(\vec{q}).$$

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. This relation is known as the "Gordon identity". Note that this is named after Walter Gordon of Klein-Gordon equation fame, and not Paul Gordan of Clebsch-Gordan coefficient fame.

(b) Using the identity

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta},$$

where $\epsilon_{12} = +1$, show that

$$\left[\overline{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_2\right]\left[\overline{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_4\right] = -\left[\overline{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_4\right]\left[\overline{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_2\right].$$

Here $\overline{u}_{1,3}$ and $u_{2,4}$ are four different spinors. This is an example of a "Fierz identity". These identities relate products of spinor bilinears to sums of products of more useful spinor bilinears.

Both the Gordon and various Fierz identities are often used in calculations of scattering amplitudes involving fermions.

Solution 1

(a) To prove the Gordon identity, we first note that

$$\overline{u}^r(\vec{p})\gamma^\mu(\gamma^\nu q_\nu - m)u^s(\vec{q}) = 0,$$

and that

$$\overline{u}^r(\vec{p})(\gamma^\nu p_\nu - m)\gamma^\mu u^s(\vec{p}) = 0.$$

Adding these together, we obtain

$$2m\overline{u}^r(\vec{p})\gamma^\mu u^s(\vec{q}) = \overline{u}^r(\gamma^\mu\gamma^\nu q_\nu + \gamma^\nu\gamma^\mu p_\nu)u^s(\vec{q}).$$

But we can write (recall lecture 10)

$$\begin{aligned}\gamma^\mu\gamma^\nu &= g^{\mu\nu} - i\sigma^{\mu\nu},\\ \gamma^\nu\gamma^\mu &= g^{\mu\nu} + i\sigma^{\nu\mu},\end{aligned}$$

so we have

$$\begin{aligned}\overline{u}^r(\vec{p})\gamma^\mu u^s(\vec{q}) &= \frac{1}{2m}\overline{u}^r\left[q_\nu\left(g^{\mu\nu} - i\sigma^{\mu\nu}\right) + p_\nu\left(g^{\mu\nu} + i\sigma^{\nu\mu}\right)\right]u^s(\vec{q})\\ &= \frac{1}{2m}\overline{u}^r\left[q^\mu + p^\mu + i\sigma^{\mu\nu}(p_\nu - q_\nu)\right]u^s(\vec{q}).\end{aligned}$$

(b) To prove this particular Fierz identity, we first note that the projection operator serves to isolate the right-handed components of the two spinors u_2 and u_4 (i.e. the corresponding Weyl spinors). Then, in the Weyl basis, the gamma matrix γ^μ are block off-diagonal. Thus the left hand side of this identity reduces to

$$\left[\overline{u}_{1,R}\sigma^\mu u_{2,R}\right]\left[\overline{u}_{3,R}\sigma_\mu u_{4,R}\right].$$

Then, applying the identity given in the question, this becomes

$$\begin{aligned}\left[\overline{u}_{1,R}\sigma^\mu u_{2,R}\right]\left[\overline{u}_{3,R}\sigma_\mu u_{4,R}\right] &= (\overline{u}_{1,R})_\alpha(\sigma^\mu)_{\alpha\beta}(u_{2,R})_\beta(\overline{u}_{3,R})_\gamma(\sigma_\mu)_{\gamma\delta}(u_{4,R})_\delta\\ &= 2\epsilon_{\alpha\gamma}(\overline{u}_{1,R})_\alpha(\overline{u}_{3,R})_\gamma\epsilon_{\beta\delta}(u_{2,R})_\beta(u_{4,R})_\delta\\ &= -2\epsilon_{\alpha\gamma}(\overline{u}_{1,R})_\alpha(\overline{u}_{3,R})_\gamma\epsilon_{\delta\beta}(u_{2,R})_\beta(u_{4,R})_\delta\\ &= -(\overline{u}_{1,R})_\alpha(\sigma^\mu)_{\alpha\delta}(u_{2,R})_\beta(\overline{u}_{3,R})_\gamma(\sigma_\mu)_{\gamma\beta}(u_{4,R})_\delta\\ &= -\left[\overline{u}_{1,R}\sigma^\mu u_{4,R}\right]\left[\overline{u}_{3,R}\sigma_\mu u_{2,R}\right]\\ &= -\left[\overline{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_4\right]\left[\overline{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_2\right]\end{aligned}$$

Question 2 — 10pts

Write 250 to 300 words[1] discussing the relationship between particles and fields in quantum field theory. You should illustrate this with at least two examples. Your response should be written in full sentences and addressed to other graduate students who have taken QFT, but don't use it regularly in their research and perhaps have forgotten some key pieces. You will be graded using the following rubric:

[1] I will count.

Aspect	Points	If you:
Physics	4	Correctly characterise: Lorentz and Poincaré groups; representations of those two groups; relation between particles and the Poincaré group, and fields and the Lorentz group.
	2	Correctly characterise most of these points, but not all; or describe all points, but miss key information.
	0	Completely misconstrue the relationships.
Examples	4	Provide at least two examples.
	2	Provide one example; or two examples, but not correctly.
	0	Give no examples
Audience	2	Correctly gauge the understanding of the audience, including defining and illustrating terms as appropriate.
	1	Give a too-technical or too-simplistic explanation.

Solution 2

The rubric is the solution here.