Quantum Field Theory I: PHYS 721
Problem Set 5: Solutions

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Overview

The questions in this problem set give you some practice at manipulating quantised operators and reinforce the close relationship between symmetries and conservation laws. There are two questions.

Question 1 **8pts**

(a) Use the expressions for the Hamiltonian and the momentum operator

$$H_0 = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} E_k a^\dagger(\vec{k})a(\vec{k}),$$
$$P^i = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} k^i a^\dagger(\vec{k})a(\vec{k})$$

for a free, real scalar field, $\phi(x)$, to show that the four-vector (H, P^i) generates spacetime translations

$$\phi(x) = e^{i(Ht-\vec{P}\cdot\vec{x})}\phi(0)e^{-i(Ht-\vec{P}\cdot\vec{x})}.$$

This is an example of a very general phenomenon: the conserved charge due to a symmetry generates the corresponding symmetry transformation on the fields.

[Hint: You will need to consider expressions of the form $e^{iH_0t}a(\vec{k})e^{-iH_0t}$.]

(b) Use this result to show that

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi(x-y)\phi(0)|0>.$$

Solution 1

(a) Following the hint given in the question, we first study

$$e^{iH_0t}a(\vec{k})e^{-iH_0t} = \sum_{n=0}^{\infty} \frac{(iH_0t)^n}{n!} a(\vec{k})e^{-iH_0t}.$$

Then, using

$$[H_0, a(\vec{k})] = -E_k a(\vec{k}),$$

we have

$$H_0 a(\vec{k}) = a(\vec{k})(H_0 - E_k).$$

This means that

$$H_0^n a(\vec{k}) = a(\vec{k})(H_0 - E_k)^n.$$

Therefore we find

$$\begin{aligned} e^{iH_0 t} a(\vec{k}) e^{-iH_0 t} &= a(\vec{k}) \sum_{n=0}^{\infty} \frac{(iH_0 t - iE_k t)^n}{n!} e^{-iH_0 t} \\ &= a(\vec{k}) e^{iH_0 t - iE_k t} e^{-iH_0 t} \\ &= a(\vec{k}) e^{-iE_k t}. \end{aligned}$$

Similar arguments lead us to

$$\begin{aligned} e^{iH_0 t} a^\dagger(\vec{k}) e^{-iH_0 t} &= a^\dagger(\vec{k}) e^{iE_k t}, \\ e^{-i\vec{P}\cdot\vec{x}} a(\vec{k}) e^{i\vec{P}\cdot\vec{x}} &= a(\vec{k}) e^{i\vec{k}\cdot\vec{x}}, \\ e^{-i\vec{P}\cdot\vec{x}} a^\dagger(\vec{k}) e^{i\vec{P}\cdot\vec{x}} &= a^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}. \end{aligned}$$

For the last two equations we have used

$$\begin{aligned} [P^i, a(\vec{k})] &= -k^i a(\vec{k}), \\ [P^i, a^\dagger(\vec{k})] &= k^i a(\vec{k}). \end{aligned}$$

Now let's take the right hand side of the equation we wish to prove. This is

$$e^{i(Ht - \vec{P}\cdot\vec{x})} \phi(0) e^{-i(Ht - \vec{P}\cdot\vec{x})} = e^{i(Ht - \vec{P}\cdot\vec{x})} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \Big(a(\vec{k}) + a^\dagger(\vec{k}) \Big) e^{-i(Ht - \vec{P}\cdot\vec{x})}.$$

But

$$e^{i(Ht - \vec{P}\cdot\vec{x})} a(\vec{k}) e^{-i(Ht - \vec{P}\cdot\vec{x})} = a(\vec{k}) e^{-i(E_k t - \vec{k}\cdot\vec{x})}$$

and

$$e^{i(Ht - \vec{P}\cdot\vec{x})} a^\dagger(\vec{k}) e^{-i(Ht - \vec{P}\cdot\vec{x})} = a^\dagger(\vec{k}) e^{i(E_k t - \vec{k}\cdot\vec{x})},$$

so we have

$$\begin{aligned} e^{i(Ht - \vec{P}\cdot\vec{x})} \phi(0) e^{-i(Ht - \vec{P}\cdot\vec{x})} &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \Big(a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \Big) \\ &= \phi(x). \end{aligned}$$

(b) First we need to note that $P^\mu|0\rangle = 0$ (where $P^\mu = (H, \vec{P})$, and that momentum operators commute $[P^\mu, P^\nu] = 0$, so that

$$e^{iP\cdot x}|0\rangle = |0\rangle.$$

Then we write

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}e^{iP\cdot y}\phi(0)e^{-iP\cdot y}|0\rangle \\ &= \langle 0|e^{-iP\cdot y}e^{iP\cdot x}\phi(0)e^{-iP\cdot x}e^{iP\cdot y}\phi(0)|0\rangle \\ &= \langle 0|e^{iP\cdot(x-y)}\phi(0)e^{-iP\cdot(x-y)}\phi(0)|0\rangle \\ &= \langle 0|\phi(x-y)\phi(0)|0\rangle.\end{aligned}$$

Question 2 [based on Peskin and Shroeder 2.2] [12]

In this question we study the quantum field theory of a complex scalar field, defined by the action

$$\widehat{S} = \int \mathrm{d}^4x \left(\partial_\mu\widehat{\phi}^*\partial^\mu\widehat{\phi} - m^2\widehat{\phi}^*\widehat{\phi}\right). \tag{1}$$

We could choose to analyse this theory by treating the real and imaginary parts of the complex field $\widehat{\phi}$ as independent dynamical variables, but it is easier to instead choose $\widehat{\phi}$ and $\widehat{\phi}^*$ as the basic independent variables.

(a) Find the conjugate momenta of $\widehat{\phi}(x)$ and $\widehat{\phi}^*(x)$ and the corresponding canonical commutation relations.

(b) Show that the Hamiltonian is

$$\widehat{H} = \int \mathrm{d}^3\vec{x} \left(\widehat{\pi}^*\widehat{\pi} + \vec{\nabla}\widehat{\phi}^* \cdot \vec{\nabla}\widehat{\phi} + m^2\widehat{\phi}^*\widehat{\phi}\right). \tag{2}$$

Determine the Heisenberg equation of motion for $\widehat{\phi}(x)$ and show that it leads to the Klein-Gordon equation.

(c) Write the Hamiltonian in terms of creation and annihilation operators[1] and show that the theory contains two sets of particles of mass m.

(d) Rewrite the conserved charge

$$\widehat{Q} = \frac{i}{2}\int \mathrm{d}^3\vec{x} \left(\widehat{\phi}^*\widehat{\pi}^* - \widehat{\pi}\widehat{\phi}\right) \tag{3}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

[1] Think about why the creation and annihilation operators were complex conjugates of each other in the real scalar field case. Does that have to be true in this case?

Solution 2

(a) The Lagrangian for the complex scalar field theory is

$$\widehat{\mathcal{L}} = \partial_\mu \widehat{\phi}^* \partial^\mu \widehat{\phi} - m^2 \widehat{\phi}^* \widehat{\phi}.$$

The conjugate momenta are

$$\widehat{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\widehat{\phi}}} = \dot{\widehat{\phi}}^*, \tag{4}$$

$$\widehat{\pi}^* = \frac{\partial \mathcal{L}}{\partial \dot{\widehat{\phi}}^*} = \dot{\widehat{\phi}}, \tag{5}$$

and their commutation relations are

$$[\widehat{\phi}(\vec{x},t), \widehat{\pi}(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}), \tag{6}$$

$$[\widehat{\phi}^*(\vec{x},t), \widehat{\pi}^*(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}), \tag{7}$$

and all others are zero.

(b) The Hamiltonian is defined as

$$\widehat{H} = \int \mathrm{d}^3\vec{x} \left(\widehat{\pi}\dot{\widehat{\phi}} + \widehat{\pi}^* \dot{\widehat{\phi}}^* - \mathcal{L} \right)$$

and we use the definitions of the conjugate momenta in Equations (4) and (5) to obtain

$$\begin{aligned} \widehat{H} &= \int \mathrm{d}^3\vec{x} \left(\dot{\widehat{\phi}}^* \dot{\widehat{\phi}} + \widehat{\pi}^*\widehat{\pi} - \partial_\mu \widehat{\phi}^* \partial^\mu \widehat{\phi} + m^2 \widehat{\phi}^* \widehat{\phi} \right) \\ &= \int \mathrm{d}^3\vec{x} \left(\widehat{\pi}^*\widehat{\pi} + \vec{\nabla}\widehat{\phi}^* \vec{\nabla}\widehat{\phi} + m^2 \widehat{\phi}^* \widehat{\phi} \right), \end{aligned} \tag{8}$$

as required.

The Heisenberg equation of motion is given by

$$i\dot{\widehat{\phi}} = [\widehat{\phi}, \widehat{H}],$$

and plugging in the Hamiltonian of Equation (8), we find

$$\begin{aligned} i\dot{\widehat{\phi}}(\vec{x},t) = &\int \mathrm{d}^3\vec{x} \Big([\widehat{\phi}(\vec{x},t), \widehat{\pi}^*(\vec{y},t)\widehat{\pi}(\vec{y},t)] + [\widehat{\phi}(\vec{x},t), \vec{\nabla}\widehat{\phi}^*(\vec{y},t)\vec{\nabla}\widehat{\phi}(\vec{y},t)] \\ &\qquad + m^2 [\widehat{\phi}(\vec{x},t), \widehat{\phi}^*(\vec{y},t)\widehat{\phi}(\vec{y},t)] \Big) \\ &= \int \mathrm{d}^3\vec{x}\, \widehat{\pi}^* [\widehat{\phi}(\vec{x},t), \widehat{\pi}(\vec{y},t)] \\ &= i \int \mathrm{d}^3\vec{x}\, \widehat{\pi}^* \delta^{(3)}(\vec{x}-\vec{y}) \\ &= i\widehat{\pi}^*(\vec{x},t) \end{aligned}$$

We also need another equation of motion

$$
\begin{aligned}
i\dot{\widehat{\pi}}^*(\vec{x},t) = &\int \mathrm{d}^3\vec{x}\Big([\widehat{\pi}^*(\vec{x},t),\widehat{\pi}^*(\vec{y},t)\widehat{\pi}(\vec{y},t)] + [\widehat{\pi}^*(\vec{x},t),\vec{\nabla}\widehat{\phi}^*(\vec{y},t)\vec{\nabla}\widehat{\phi}(\vec{y},t)] \\
&\quad + m^2[\widehat{\pi}^*(\vec{x},t),\widehat{\phi}^*(\vec{y},t)\widehat{\phi}(\vec{y},t)]\Big) \\
&= \int \mathrm{d}^3\vec{x}\,\Big([\widehat{\pi}^*(\vec{x},t),(-\vec{\nabla}^2\widehat{\phi}^*(\vec{y},t))]\widehat{\phi}(\vec{y},t) + m^2[\widehat{\pi}^*(\vec{x},t),\widehat{\phi}^*(\vec{y},t)]\widehat{\phi}(\vec{y},t)\Big) \\
&= \int \mathrm{d}^3\vec{x}\,(-i)\delta^{(3)}(\vec{x}-\vec{y})(-\vec{\nabla}^2 + m^2)\widehat{\phi}(\vec{y},t) \\
&= i(\vec{\nabla}^2 - m^2)\widehat{\phi}(\vec{x},t).
\end{aligned}
$$

Therefore

$$
\ddot{\widehat{\phi}}(\vec{x},t) = \dot{\widehat{\pi}}^*(\vec{x},t) = (\vec{\nabla}^2 - m^2)\widehat{\phi}(\vec{x},t),
$$

which is the Klein-Gordon equation for $\widehat{\phi}$.

(c) First we must express the fields in terms of annihilation and creation operators. Since we do not require the fields to be real, we no longer restrict the annihilation and creation operators to be complex conjugates of each other. Thus we write

$$
\begin{aligned}
\widehat{\phi}(x) &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left(\widehat{a}(\vec{k})e^{-ik\cdot x} + \widehat{b}^\dagger(\vec{k})e^{ik\cdot x}\right), \\
\widehat{\phi}^*(x) &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left(\widehat{a}^\dagger(\vec{k})e^{ik\cdot x} + \widehat{b}(\vec{k})e^{-ik\cdot x}\right).
\end{aligned}
$$

Now we plug these expressions into our Hamiltonian, Equation (8):

$$
\begin{aligned}
\widehat{H} &= \int \mathrm{d}^3\vec{x} \left(\widehat{\pi}^* \widehat{\pi} + \vec{\nabla}\widehat{\phi}^* \vec{\nabla}\widehat{\phi} + m^2 \widehat{\phi}^* \widehat{\phi} \right) \\
&= \int \mathrm{d}^3\vec{x} \int \frac{\mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{p}}{(2\pi)^6} \Bigg\{ \frac{iE_k}{\sqrt{2E_k}} \left(\widehat{a}^\dagger(\vec{k}) e^{ik\cdot x} - \widehat{b}(\vec{k}) e^{-ik\cdot x} \right) \frac{iE_p}{\sqrt{2E_p}} \left(-\widehat{a}(\vec{p}) e^{-ip\cdot x} + \widehat{b}^\dagger(\vec{p}) e^{ip\cdot x} \right) \\
&\qquad + \frac{(-ik^i)}{\sqrt{2E_k}} \left(\widehat{a}^\dagger(\vec{k}) e^{ik\cdot x} + \widehat{b}(\vec{k}) e^{-ik\cdot x} \right) \frac{ip^i}{\sqrt{2E_p}} \left(\widehat{a}(\vec{p}) e^{-ip\cdot x} - \widehat{b}^\dagger(\vec{p}) e^{ip\cdot x} \right) \\
&\qquad + m^2 \frac{1}{\sqrt{2E_k}} \left(\widehat{a}^\dagger(\vec{k}) e^{ik\cdot x} + \widehat{b}(\vec{k}) e^{-ik\cdot x} \right) \frac{1}{\sqrt{2E_p}} \left(\widehat{a}(\vec{p}) e^{-ip\cdot x} + \widehat{b}^\dagger(\vec{p}) e^{ip\cdot x} \right) \Bigg\} \\
&= \int \mathrm{d}^3\vec{x} \int \frac{\mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{p}}{(2\pi)^6} \frac{1}{\sqrt{4E_k E_p}} \\
&\quad \times \Bigg\{ (E_k E_p + \vec{k}\cdot\vec{p} + m^2) \left(\widehat{a}^\dagger(\vec{k}) \widehat{a}(\vec{p}) e^{i(k-p)\cdot x} + \widehat{b}(\vec{k}) \widehat{b}^\dagger(\vec{p}) e^{-i(k-p)\cdot x} \right) \\
&\qquad - (E_k E_p + \vec{k}\cdot\vec{p} - m^2) \left(\widehat{a}^\dagger(\vec{k}) \widehat{b}^\dagger(\vec{p}) e^{i(k+p)\cdot x} + \widehat{b}(\vec{k}) \widehat{a}(\vec{p}) e^{-i(k+p)\cdot x} \right) \Bigg\} \\
&= \int \frac{\mathrm{d}^3\vec{k}\,\mathrm{d}^3\vec{p}}{(2\pi)^6} \frac{1}{2\sqrt{E_k E_p}} \\
&\quad \times \Bigg\{ (E_k E_p + \vec{k}\cdot\vec{p} + m^2) \left(\widehat{a}^\dagger(\vec{k}) \widehat{a}(\vec{p}) e^{i(E_k - E_p)t} + \widehat{b}(\vec{k}) \widehat{b}^\dagger(\vec{p}) e^{-i(E_k - E_p)t} \right) \\
&\qquad \times (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) \\
&\quad - (E_k E_p + \vec{k}\cdot\vec{p} - m^2) \left(\widehat{a}^\dagger(\vec{k}) \widehat{b}^\dagger(\vec{p}) e^{i(E_k + E_p)t} + \widehat{b}(\vec{k}) \widehat{a}(\vec{p}) e^{-i(E_k + E_p)t} \right) \\
&\qquad \times (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{p}) \Bigg\} \\
&= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \frac{1}{2E_k} 2E_k^2 \left(\widehat{a}^\dagger(\vec{k}) \widehat{a}(\vec{k}) + \widehat{b}(\vec{k}) \widehat{b}^\dagger(\vec{k}) \right) \\
&= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} E_k \left(\widehat{a}^\dagger(\vec{k}) \widehat{a}(\vec{k}) + \widehat{b}^\dagger(\vec{k}) \widehat{b}(\vec{k}) + [\widehat{b}(\vec{k}), \widehat{b}^\dagger(\vec{k})] \right).
\end{aligned}
$$

In the second line we implicitly sum over spatial indices i, and we use $E_k^2 = |\vec{k}|^2 + m^2$ in the fourth line.

Normal ordering this Hamiltonian gives

$$
:\widehat{H}: = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} E_k \left(\widehat{a}^\dagger(\vec{k}) \widehat{a}(\vec{p}) + \widehat{b}^\dagger(\vec{k}) \widehat{b}(\vec{k}) \right).
$$

Thus we have two sets of particles (because there are two different operator combinations that, in a hand-wavy way, both correspond to a number operator for their respective particle species), each with the same mass (because both satisfy the relativistic dispersion relation $E_k^2 = |\vec{k}|^2 + m^2$).

(d) For the conserved charge, we follow an analogous procedure as we did for the Hamiltonian. Thus we obtain

$$
\begin{aligned}
\widehat{Q} &= \frac{i}{2}\int \mathrm{d}^3\vec{x}\left(\widehat{\phi}^*\widehat{\pi}^* - \widehat{\pi}\widehat{\phi}\right) \\
&= \frac{i}{2}\int \mathrm{d}^3\vec{x}\int \frac{\mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}}{(2\pi)^6}\Bigg\{\frac{1}{\sqrt{2E_k}}\left(\widehat{a}^\dagger(\vec{k})e^{ik\cdot x} + \widehat{b}(\vec{k})e^{-ik\cdot x}\right)\frac{iE_p}{\sqrt{2E_p}}\left(-\widehat{a}(\vec{p})e^{-ip\cdot x} + \widehat{b}^\dagger(\vec{p})e^{ip\cdot x}\right) \\
&\qquad - \frac{iE_k}{\sqrt{2E_k}}\left(\widehat{a}^\dagger(\vec{k})e^{ik\cdot x} - \widehat{b}(\vec{k})e^{-ik\cdot x}\right)\frac{1}{\sqrt{2E_p}}\left(\widehat{a}(\vec{p})e^{-ip\cdot x} + \widehat{b}^\dagger(\vec{p})e^{ip\cdot x}\right)\Bigg\} \\
&= \frac{i}{2}\int \mathrm{d}^3\vec{x}\int \frac{\mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}}{(2\pi)^6}\frac{1}{\sqrt{4E_kE_p}} \\
&\quad \times \Bigg\{(iE_p + iE_k)\left(-\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{p})e^{i(k-p)\cdot x} + \widehat{b}(\vec{k})\widehat{b}^\dagger(\vec{p})e^{-i(k-p)\cdot x}\right) \\
&\qquad + (iE_p - iE_k)\left(\widehat{a}^\dagger(\vec{k})\widehat{b}^\dagger(\vec{p})e^{i(k+p)\cdot x} - \widehat{b}(\vec{k})\widehat{a}(\vec{p})e^{-i(k+p)\cdot x}\right)\Bigg\} \\
&= \frac{i}{2}\int \frac{\mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}}{(2\pi)^6}\frac{1}{2\sqrt{E_kE_p}} \\
&\quad \times \Bigg\{i(E_p + E_k)\left(-\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{p})e^{i(E_k-E_p)t} + \widehat{b}(\vec{k})\widehat{b}^\dagger(\vec{p})e^{-i(E_k-E_p)t}\right)(2\pi)^3\delta^{(3)}(\vec{k}-\vec{p}) \\
&\qquad + i(E_p - E_k)\left(\widehat{a}^\dagger(\vec{k})\widehat{b}^\dagger(\vec{p})e^{i(E_k+E_p)t} + \widehat{b}(\vec{k})\widehat{a}(\vec{p})e^{-i(E_k+E_p)t}\right)(2\pi)^3\delta^{(3)}(\vec{k}+\vec{p})\Bigg\} \\
&= \frac{i}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k}(-2i)E_k\left(\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}) - \widehat{b}(\vec{k})\widehat{b}^\dagger(\vec{k})\right) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\left(\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}) - \widehat{b}^\dagger(\vec{k})\widehat{b}(\vec{k}) - [\widehat{b}(\vec{k}),\widehat{b}^\dagger(\vec{k})]\right).
\end{aligned}
$$

Now we normal-order again, and we find

$$
:\widehat{Q}: = \frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\left(\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}) - \widehat{b}^\dagger(\vec{k})\widehat{b}(\vec{k})\right).
$$

We can already see that this is going to tell us that there are two particles, with charge

$\pm 1/2$. We confirm this by calculating the commutators

$$\begin{aligned}
[:\widehat{Q}:, \widehat{a}^\dagger(\vec{p})] &= \frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\left([\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}), \widehat{a}^\dagger(\vec{p})] - [\widehat{b}^\dagger(\vec{k})\widehat{b}(\vec{k}), \widehat{a}^\dagger(\vec{p})]\right) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}[\widehat{a}(\vec{k}), \widehat{a}(\vec{p})]\widehat{a}^\dagger(\vec{k}) \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}(2\pi)^3\delta^{(3)}(k-p)\widehat{a}^\dagger(\vec{k}) \\
&= \frac{1}{2}\widehat{a}^\dagger(\vec{p}).
\end{aligned}$$

Thus $\widehat{a}^\dagger(\vec{p})$ creates a particle of charge $+1/2$. Using the same logic, we find

$$[:\widehat{Q}:, \widehat{b}^\dagger(\vec{p})] = -\frac{1}{2}\widehat{b}^\dagger(\vec{p}),$$

so $\widehat{b}^\dagger(\vec{p})$ creates a particle of charge $-1/2$.

Comment Note that the equivalent form of the charge conjugation operator is

$$
\begin{aligned}
\widehat{\mathcal{Q}} &= \frac{i}{2}\int \mathrm{d}^3\vec{x}\left(\widehat{\pi}^*\widehat{\phi}^* - \widehat{\pi}\widehat{\phi}\right) \\
&= \frac{i}{2}\int \mathrm{d}^3\vec{x}\int \frac{\mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}}{(2\pi)^6}\Big\{ \frac{iE_p}{\sqrt{2E_p}}\left(-\widehat{a}(\vec{p})e^{-ip\cdot x} + \widehat{b}^\dagger(\vec{p})e^{ip\cdot x}\right)\frac{1}{\sqrt{2E_k}}\left(\widehat{a}^\dagger(\vec{k})e^{ik\cdot x} + \widehat{b}(\vec{k})e^{-ik\cdot x}\right) \\
&\qquad - \frac{iE_k}{\sqrt{2E_k}}\left(\widehat{a}^\dagger(\vec{k})e^{ik\cdot x} - \widehat{b}(\vec{k})e^{-ik\cdot x}\right)\frac{1}{\sqrt{2E_p}}\left(\widehat{a}(\vec{p})e^{-ip\cdot x} + \widehat{b}^\dagger(\vec{p})e^{ip\cdot x}\right)\Big\} \\
&= \frac{i}{2}\int \mathrm{d}^3\vec{x}\int \frac{\mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}}{(2\pi)^6}\frac{1}{\sqrt{4E_kE_p}} \\
&\quad \times \Big\{ iE_p\left(-\widehat{a}(\vec{p})\widehat{a}^\dagger(\vec{k})e^{i(k-p)\cdot x} + \widehat{b}^\dagger(\vec{p})\widehat{b}(\vec{k})e^{-i(k-p)\cdot x}\right) \\
&\qquad - iE_k\left(\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{p})e^{i(k-p)\cdot x} - \widehat{b}(\vec{k})\widehat{b}^\dagger(\vec{p})e^{-i(k-p)\cdot x}\right) \\
&\qquad + iE_p\left(\widehat{b}^\dagger(\vec{p})\widehat{a}^\dagger(\vec{k})e^{i(k+p)\cdot x} - \widehat{a}(\vec{p})\widehat{b}(\vec{k})e^{-i(k+p)\cdot x}\right) \\
&\qquad - iE_k\left(\widehat{a}^\dagger(\vec{k})\widehat{b}^\dagger(\vec{p})e^{i(k+p)\cdot x} - \widehat{b}(\vec{k})\widehat{a}(\vec{p})e^{-i(k+p)\cdot x}\right)\Big\} \\
&= \frac{i}{2}\int \frac{\mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}}{(2\pi)^6}\frac{1}{\sqrt{4E_kE_p}} \\
&\quad \times \Big\{ \Big[-i\left(E_p\widehat{a}(\vec{p})\widehat{a}^\dagger(\vec{k}) + E_k\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{p})\right)e^{i(E_k-E_p)t} \\
&\qquad\qquad + i\left(E_p\widehat{b}^\dagger(\vec{p})\widehat{b}(\vec{k}) + E_k\widehat{b}(\vec{k})\widehat{b}^\dagger(\vec{p})\right)e^{-i(E_k-E_p)t}\Big](2\pi)^3\delta^{(3)}(\vec{k}-\vec{p}) \\
&\qquad + i(E_p - E_k)\left(\widehat{b}^\dagger(\vec{p})\widehat{a}^\dagger(\vec{k})e^{i(E_k+E_p)t} - \widehat{a}(\vec{p})\widehat{b}(\vec{k})e^{-i(E_k+E_p)t}\right)(2\pi)^3\delta^{(3)}(\vec{k}+\vec{p})\Big\}.
\end{aligned}
$$

Now, if $\mathbf{k} = \pm\mathbf{p}$, then $E_k = E_p$ and the second set of terms vanish, while the first set simplify

$$
\begin{aligned}
\widehat{\mathcal{Q}} &= \frac{i}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{i}{2E_k}E_k\Big[-\widehat{a}(\vec{k})\widehat{a}^\dagger(\vec{k}) - \widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}) + \widehat{b}^\dagger(\vec{k})\widehat{b}(\vec{k}) + \widehat{b}(\vec{k})\widehat{b}^\dagger(\vec{k})\Big] \\
&= \frac{1}{4}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\Big[\widehat{a}(\vec{k})\widehat{a}^\dagger(\vec{k}) + \widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}) - \widehat{b}^\dagger(\vec{k})\widehat{b}(\vec{k}) - \widehat{b}(\vec{k})\widehat{b}^\dagger(\vec{k})\Big].
\end{aligned}
$$

Now we normal-order again, and we find

$$
:\widehat{\mathcal{Q}}: = \frac{1}{2}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\left(\widehat{a}^\dagger(\vec{k})\widehat{a}(\vec{k}) - \widehat{b}^\dagger(\vec{k})\widehat{b}(\vec{k})\right),
$$

which is exactly the result we obtained for $:\widehat{Q}:$.