Quantum Field Theory I: PHYS 721
Problem Set 9: Solutions

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There are three questions in this problem set. Answer **two** questions, for a total of twenty points: answer question 1 and **either** question 2 **or** question 3. There are no bonus points for answering more than two questions, although I encourage you to do so for your own understanding.

The first question in this problem set will illustrate the process of calculating an experimental observable, in this case the decay rate, from Feynman diagrams. The second question investigates the role of symmetries in correlation functions of gauge fields. The third question helps you to synthesise the material we have studied throughout the course.

Question 1 [10]

Recall the scalar Yukawa theory from previous problem sets:

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - M^2 \psi^* \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - g\psi^* \psi \phi.$$

Here ψ is a complex scalar field and ϕ is a real scalar field. Consider again the case where the "meson" mass, m, is greater than (twice) the "nucleon" (or "antinucleon") mass, M, which allows for the meson decay process $\phi \to \psi\psi^*$ (see Problem Set 8).

In the last problem set, you wrote down the Feynman diagram for this process of meson decay. In this question, you will now follow the rest of the steps required to evaluate the total decay rate for this process.

(a) Use the Feynman rules for this theory to write down an expression for the leading order (tree-level) contribution to the invariant matrix element, $\mathcal{M}(P \to \{p_f\})$, for meson decay, $\phi \to \psi\psi^*$.

(b) Work in the rest frame of the heavy meson, and by summing/integrating over all final-state phase space, use the differential decay rate

$$\mathrm{d}\Gamma = \frac{1}{2m} \prod_{f=1}^{n} \frac{\mathrm{d}^3 \vec{p}_f}{(2\pi)^3} \frac{1}{2E_f} (2\pi)^4 \delta^{(4)}(P - \sum_f p_f) |\mathcal{M}(P \to \{p_f\})|^2,$$

where f labels all the final state particles, to show that the total decay rate, Γ, is given by

$$\Gamma = \frac{g^2}{16\pi} \frac{\sqrt{1 - 4M^2/m^2}}{m}.$$

Solution 1

(a) The invariant matrix element is

$$i\mathcal{M} = -ig.$$

(b) To obtain the total decay rate, Γ, we integrate the differential decay rate over the two final state momenta

$$\Gamma = \frac{1}{2m}\int \frac{\mathrm{d}^3\vec{q}_1}{(2\pi)^3}\frac{1}{2E_{q_1}}\int \frac{\mathrm{d}^3\vec{q}_2}{(2\pi)^3}\frac{1}{2E_{q_2}}(2\pi)^4\delta^{(4)}(P-q_1-q_2)|\mathcal{M}(P\to q_1,q_2)|^2.$$

In the rest frame of the heavy meson, we have $E=m$ and $\vec{P}=0$. Then the delta function becomes

$$\begin{aligned}\delta^{(4)}(P-q_1-q_2) &= \delta^{(3)}(\vec{q}_1+\vec{q}_2)\delta(E-E_{q_1}-E_{q_2}) \\ &= \delta^{(3)}(\vec{q}_1+\vec{q}_2)\delta(E-2E_{q_1}),\end{aligned}$$

where I have used $E_{-q_1}=E_{q_1}$. We can now carry out the $\vec{q}_2$ integral to give

$$\begin{aligned}\Gamma &= \frac{g^2}{2m}\int \frac{\mathrm{d}^3\vec{q}_1}{(2\pi)^3}\frac{1}{4E_{q_1}^2}(2\pi)\delta(E-2E_{q_1}) \\ &= \frac{g^2}{2m}\int \mathrm{d}\Omega_2\int \frac{q_1^2\mathrm{d}q_1}{(2\pi)^3}\frac{1}{4E_{q_1}^2}(2\pi)\delta(E-2E_{q_1}).\end{aligned}$$

Now we use the variable substitution $E_{q_1}^2=q_1^2+m^2$, so

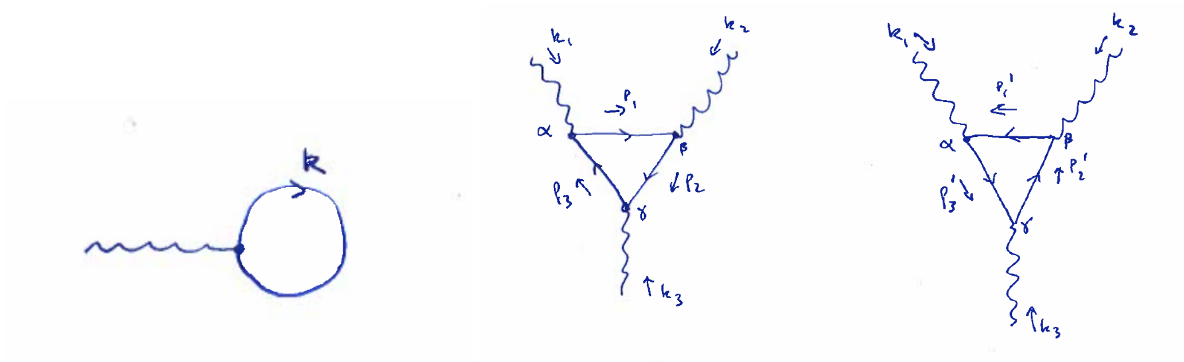
$$E_{q_1}\mathrm{d}E_{q_1}=q_1\mathrm{d}q_1,$$

and we have

$$\begin{aligned}\Gamma &= \frac{g^2}{2m}(4\pi)\int \frac{E_{q_1}\mathrm{d}E_{q_1}}{(2\pi)^3}\frac{\sqrt{E_{q_1}^2-M^2}}{4E_{q_1}^2}(2\pi)\delta(m-2E_{q_1}) \\ &= \frac{g^2}{8m\pi}\int \mathrm{d}E_q\frac{\sqrt{E_q^2-M^2}}{E_q}\frac{1}{2}\delta(m/2-E_q) \\ &= \frac{g^2}{16m\pi}\frac{\sqrt{m^2/4-M^2}}{m/2} \\ &= \frac{g^2}{16\pi}\frac{\sqrt{1-\mu^2}}{m},\end{aligned}$$

where $\mu=2M/m<1$ for physical decays.

Figure 1: Feynman diagrams that contribute to the photon one- and three-point functions in QED.



Question 2 [Choose either this or question 3] [10]

This question introduces you to one-loop diagrams, and demonstrates that sometimes symmetry properties are sufficient to deduce results in QFT. Recall that, in the presence of loops in Feynman diagrams, there is an extra Feynman rule: "integrate over any loop momenta". A full treatment of the consequences of this Feynman rule will be given in Quantum Field Theory II and you will not need to evaluate any one-loop integrals to answer these questions!

(a) Evaluate the one-fermion-loop contribution to the photon one-point function, $\langle\Omega|A_\mu|\Omega\rangle$, and show that it vanishes. You shouldn't need to explicitly carry out the integral to show that the result is zero.

(b) Now consider the photon three-point function, $\langle\Omega|A_\mu A_\nu A_\rho|\Omega\rangle$, and write down the two one-loop diagrams that contribute to this vacuum expectation value. Show that individually these contributions are nonzero, but that their sum is zero. Hint: you will need to carefully consider the trace identities for gamma matrices, but you do not need to evaluate any integrals.

(c) By considering the properties under charge transformation of photon n-point functions, show that these functions vanish if n is odd. Note that this result is true nonperturbatively and you should not need to assume a perturbative expansion.

Solution 2

I show the relevant diagrams for the one- and three-point functions of the photon in QED in Figure 1.

(a) The one-point function is proportional to

$$\begin{aligned}
\operatorname{Tr}\gamma^\mu \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} &= \operatorname{Tr}\gamma^\mu\gamma^\nu \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\nu}{k^2-m^2+i\epsilon} \\
&= 4g^{\mu\nu} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\nu}{k^2-m^2+i\epsilon} \\
&= 4 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu}{k^2-m^2+i\epsilon}.
\end{aligned}$$

But this integral is odd under $k \to -k$, so it vanishes.

(b) The first contribution to the three-point function is proportional to

$$I_1 = \operatorname{Tr}\left(\gamma^\alpha \frac{\not{p}_1+m}{p_1^2-m^2+i\epsilon}\gamma^\beta \frac{\not{p}_2+m}{p_2^2-m^2+i\epsilon}\gamma^\gamma \frac{\not{p}_3+m}{p_3^2-m^2+i\epsilon}\right),$$

where momentum conservation ensures that

$$\begin{aligned}
p_1 + k_2 &= p_2, \\
p_2 + k_3 &= p_3, \\
p_3 + k_1 &= p_1.
\end{aligned}$$

The nonzero terms include exactly either zero or two factors of the mass, as the other contributions include a trace over an odd number of gamma matrices. These remaining terms are nonzero in general.

The second diagram is given by

$$I_2 = \operatorname{Tr}\left(\gamma^\alpha \frac{\not{q}_3+m}{q_3^2-m^2+i\epsilon}\gamma^\gamma \frac{\not{q}_2+m}{q_2^2-m^2+i\epsilon}\gamma^\beta \frac{\not{q}_1+m}{q_1^2-m^2+i\epsilon}\right),$$

where now we have

$$\begin{aligned}
q_2 + k_2 &= q_1, \\
q_3 + k_3 &= q_2, \\
q_1 + k_1 &= q_3.
\end{aligned}$$

But clearly $q_i = -p_i$, so we deduce

$$\begin{aligned}
p_2 &= p_1 + k_2, \\
p_3 &= p_2 + k_3, \\
p_1 &= p_3 + k_1.
\end{aligned}$$

These are exactly the momentum conservation relations we had for the first diagram. Thus we can write

$$\begin{aligned}
I_3 = {}& \operatorname{Tr}\left(\gamma^\alpha \frac{-\not{p}_3+m}{p_3^2-m^2+i\epsilon}\gamma^\gamma \frac{-\not{p}_2+m}{p_2^2-m^2+i\epsilon}\gamma^\beta \frac{-\not{p}_1+m}{p_1^2-m^2+i\epsilon}\right) \\
= {}& -\operatorname{Tr}\left(\frac{\not{p}_3-m}{p_3^2-m^2+i\epsilon}\gamma^\gamma \frac{\not{p}_2-m}{p_2^2-m^2+i\epsilon}\gamma^\beta \frac{\not{p}_1-m}{p_1^2-m^2+i\epsilon}\gamma^\alpha\right) \\
= {}& -\operatorname{Tr}\left(\gamma^\alpha \frac{\not{p}_1-m}{p_1^2-m^2+i\epsilon}\gamma^\beta \frac{\not{p}_2-m}{p_2^2-m^2+i\epsilon}\gamma^\gamma \frac{\not{p}_3-m}{p_3^2-m^2+i\epsilon}\right)
\end{aligned}$$

Again the only nonvanishing terms include zero or two powers of the mass. These terms have the opposite sign to the corresponding terms in I_1, so the sum vanishes.

(c) The interaction term in QED, which couples photons and fermions, is proportional to $\overline{\psi}\gamma^\mu A_\mu \psi$. We already know that charge conjugation acts on the vector bilinear as

$$\widehat{C}\overline{\psi}\gamma^\mu\psi\widehat{C}^{-1} = -\overline{\psi}\gamma^\mu\psi.$$

For QED to be charge-conjugation invariant, we must also have

$$\widehat{C}A_\mu\widehat{C}^{-1} = -A_\mu.$$

Applying this logic to the photon n-point function, Γ_n, we have

$$\begin{aligned}
\Gamma_n &= \langle\Omega|A_{\mu_1}\cdots A_{\mu_n}|\Omega\rangle \\
&= \langle\Omega|A_{\mu_1}\widehat{C}^{-1}\widehat{C}\cdots\widehat{C}^{-1}\widehat{C}A_{\mu_n}\widehat{C}^{-1}\widehat{C}|\Omega\rangle \\
&= \langle\Omega|\widehat{C}A_{\mu_1}\widehat{C}^{-1}\widehat{C}\cdots\widehat{C}^{-1}\widehat{C}A_{\mu_n}\widehat{C}^{-1}|\Omega\rangle \\
&= (-1)^n\langle\Omega|A_{\mu_1}\cdots A_{\mu_n}|\Omega\rangle,
\end{aligned}$$

where I've used

$$\widehat{C}|\Omega\rangle = |\Omega\rangle.$$

Thus we get

$$\Gamma_n = (-1)^n\Gamma_n$$

under charge conjugation. So $\Gamma_n = 0$ for n odd.

Question 3 [Choose either this or question 2] **[10]**

Write 300 to 500 words[1] summarising and synthesising the topics covered in this course. As a reminder, the course has been split into four sections: particles and fields; free field theory; interacting field theory; scattering in QED. You should illustrate the ideas from each section, and the connections between them, with examples where possible. Your response should be written in full sentences and addressed to graduate students considering taking the course (that is, you should try to relate the content of the course to concepts that gradute students at the end of their first year would be familiar with). Do not include equations unless strictly necessary. You may include diagrams illustrating relationships between concepts or equations if it helps clarify your presentation. You will be graded using the following rubric:

Aspect	**Points**	**If you:**
Physics	4	Correctly: characterise each section of the course; relate ideas in each section; discuss most (if not all of)–spin and statistics, role of symmetries, particles and Poincaré group, fields and Lorentz group, properties of different types of particles and types of field, free field solutions, relating scattering observables to quantities calculable in QFT (and the chain of logic involved), QED scattering processes.
	2	Correctly characterise most of these points, but not all; or describe all points, but miss key information.
	0	Completely misconstrue the relationships.
Examples	4	Provide examples throughout.
	2	Provide one or two examples, but not consistently throughout.
	0	Give no examples
Audience	2	Correctly gauge the understanding of the audience, including defining and illustrating terms as appropriate.
	1	Give a too-technical or too-simplistic explanation.

Solution 3

The rubric is the solution.

A diagram illustrating my own view of (most of) the relationships is in Figure 2.

[1] I will probably not count.

Figure 2: A flowchart illustrating the connection between topics in the course.